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Of the non-existence of BLD mappings between manifolds

$\mathbb{R}$ami Luisto
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Abstract

In this thesis we study existence of *bounded length distortion* (*BLD*) mappings between manifolds by mimicing the proof of a Varopoulos type result. The restraints mimic the volume-growth invariants of the case of quasiregular maps between Riemannian manifolds. Our basic method is to construct invariants from the 'coarse volumes' of covering spaces, which can be in some situations compared with the combined growth of the original manifold and its fundamental group.

University of Helsinki

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1 Introduction

The main results of this thesis are mainly motivated by results of Jormakka [Jor88] and [VSCC92] that state the following:

Theorem A. *Let $\mathcal{N}$ be compact orientable n-dimensional Riemannian manifold. If the fundamental group $\Pi_1(\mathcal{N})$ of the manifold $\mathcal{N}$ is at least of polynomial growth of order $d > n$, then there exists no quasiregular mapping from $\mathbb{R}^n$ to $\mathcal{N}$.*

This type of result is sometimes called *a Varopoulos type result.* The standard proofs given (see for example [VSCC92, theorem X.5.1, p. 146]) rely heavily on the analytical structure of quasiregular mappings, which is something we wish in this thesis to avoid. We will instead find ways to imitate the somewhat hidden idea of isoperimetric inequalities used in these proofs.

Quasiregular mappings were originally called with the maybe more descriptive name of *bounded distortion* mappings. What we shall focus on this thesis are mappings called bounded *length* distortion mappings. These are defined as follows. (Basic properties will be given in section 5.)

Definition. A continuous open discrete mapping $f\colon \mathcal{M} \to \mathcal{N}$ between two manifolds with path-length -structure is said to be a *bounded length distortion*- or a BLD mapping, if there exists a constant L such that

$$L^{-1}\ell(\gamma) \leq \ell(f \circ \gamma) \leq L\ell(\gamma)$$

for any rectifiable path $\gamma\colon [0,1] \to \mathcal{M}$. In the case of Riemannian manifolds BLD mappings appear as a special case of quasiregular mappings, but they can be defined on any manifold with a path-length structure. A path-length structure will be defined in section 4.1 but in essence a path-length structure on a metric space X is a collection of paths together with a length functional such that any two points in X can be connected with a path with length arbitrarily close to the distance of these two points.

The tools that we use are very geometrical and our spirit is that of coarse methods. Coarse geometry can be seen to be the study of co-local geometric properties. Much of the spirit of this thesis, especially its use of coarse methods, is inherited from the work of Mischa Gromov, especially from [Gro99]. Our main invariant will be the growth rate of a metric space that we will define in section 3.2. This will be a coarse concept and our main tools will be different classes of mappings that preserve some coarse properties of a space. With growth rate and some suitable classes of mappings we can transform the ideas of volume growth of Riemannian manifolds in to concepts applicable in the continuous and discrete setting.

We will spend a lot of time with growth rate, and of any mathematical property it is natural to ask under what kinds of maps is it preserved. We shall study in this thesis four classes of mappings that are connected to this question; BLD mappings (properly discussed in section 5), coarse quasi-isometries (defined in section 3.1), Lipschitz quotient maps and coarse Lipschitz quotient maps (both defined in section 3.4). We will in fact see that BLD mappings are Lipschitz quotients and that coarse quasi-isometries and Lipschitz quotient mappings are both coarse Lipschitz quotient mappings. Coarse quasi-isometries will be mappings that preserve growth rate in the best possible way, and we will use them on several occasions to conjugate our results between spaces without obstructing growth concepts. BLD mappings do not preserve growth rate completely, but are able at least in the case of manifolds to take into consideration the growth rate of of the fundamental group as well as the growth of the manifold. Coarse Lipschitz quotient mappings are, as the name suggests, coarse and more general versions of Lipschitz quotient mappings. They do not give us an equivalence relation as coarse quasi-isometries do, but they have other good properties. Most important properties of (coarse) Lipschitz quotient mappings or us are that (coarse) Lipschitz quotient mappings cannot increase growth rate in an essential manner and that BLD mappings are always Lipschitz quotient mappings when the domain is complete. The relation between coarse quasi-isometries and coarse Lipschitz quotients can be seen to be in essence the relation that exists generally between isomorphisms and morphisms. We even use these two in this manner. For example when showing the growth rate conservation property of coarse quasi-isometries in section 3.4 we basically show that the coarse quasi-isometry and its 'inverse' are both coarse Lipschitz quotients and thus cannot increase growth rate in either direction.

The coarse methods cannot talk about any local properties in a natural manner. For the definition and basic properties of BLD mappings we will however need to use lots of concepts relying on local concepts. The natural domain for a BLD mapping will be a complete manifold, but we will need to bind the geometry of our manifolds in order to get the best possible results. First of all, our concept of growth rate is hard to define uniquely and more mechanical to use if the volume of our manifold grows in a superexponential manner. In section 3.2.2 we will give a restriction by requiring a condition of weak doublingness and note that this condition is filled by Riemannian manifolds with Ricci curvature bounded from below. Other bounds concern the minimal and maximal size of non null-homotopic loops. These requirements will give rise to good properties of the fundamental group of our manifold. (It will be finitely generated and as such weakly doubling for example.) Also as BLD mappings are defined by how they act on rectifiable paths we must study how to find, use and transfer path-length structures.

Our main results will give non-existence of BLD mappings in a more wider setting than what is given in a standard Varopoulos type result by showing that BLD mappings lift to BLD mappings and that they cannot increase growth rates. What we want to prove will essentially be the following: *(Theorem B corresponds to theorem 5.17 when combined with results from section 5.5 that generalize our results to quasiconvex manifolds.)*

Theorem B. *Let $\mathcal{M}$ and $\mathcal{N}$ be quasiconvex manifolds and assume $\mathcal{M}$ is complete. If there exists a BLD map $f\colon \mathcal{M} \to \mathcal{N}$, then*

$$\mathrm{Ord}(\widetilde{\mathcal{M}}) \geq \mathrm{Ord}(\widetilde{\mathcal{N}}) \quad \textit{and} \quad \mathrm{Ord}(\mathcal{M}) \geq \mathrm{Ord}(\mathcal{N})$$

where $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{N}}$ are the universal covers of $\mathcal{M}$ and $\mathcal{N}$, respectively.

Corollary C. *Let $\mathcal{N}$ be compact quasiconvex n-dimensional manifold. The fundamental group of $\mathcal{N}$ is finitely generated and coarsely quasi-isometric to the universal cover of $\mathcal{N}$. If the growth rate of $\Pi_1(\mathcal{N})$ is strictly greater than polynomial rate of order n, there exists no BLD mapping $f\colon \mathbb{R}^n \to \mathcal{N}$.*

2 Notation, basic definitions and preliminary notions

We begin by agreeing on our notation. The composition of mappings $f\colon A \to B$ and $g\colon B \to C$ is written $g \circ f$, so $(g \circ f)(x) = g(f(x))$. The composition of *paths*, however, is written as follows. If $\alpha, \beta\colon [0,1] \to X$, with $\alpha(1) = \beta(0)$, then the composition of these paths is denoted by $\alpha * \beta$ and defined to be

$$\alpha * \beta\colon [0,1] \to X, \quad \alpha * \beta(t) = \begin{cases} \alpha(2t), & \text{when } t \in [0, \frac{1}{2}] \\ \beta(2t-1), & \text{when } t \in [\frac{1}{2}, 1] \end{cases}$$

If we are composing a large number of paths we may omit the asterisk and denote $\alpha\beta := \alpha * \beta$ if it clarifies the notation. The image of a path as a mapping $\gamma\colon [0,1] \to X$ is sometimes denoted $|\gamma| := \mathrm{Im}(\gamma)$.

A path $\gamma\colon [0,1] \to X$ with $\gamma(0) = \gamma(1)$ is called a *loop*. We will often need paths connecting given two points in a space. If x and y are two points in a path-connected topological space X, we denote by $\gamma\colon x \curvearrowright y$ a continuous path $\gamma\colon [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Suppose that in addition the space in question has a metric d and a well defined path-length ℓ for paths (see section 4.1 for details). If we then have that $\ell(\gamma) = d(x,y)$, we denote $\gamma\colon x \underset{g}{\curvearrowright} y$. Such a length-minimizing path is called a geodesic.

If we have a function $f\colon X \to Y$ and subsets $A \subset X$, $B \subset Y$ such that $f[A] \subset B$ we denote $f\colon (X,A) \to (Y,B)$. If the subset in question consists of a single point $x_0 \in X$ we call the pair $(X, \{x_0\})$ a set or a space with a fixed point and denote it by (X, x_0). A mapping $f\colon X \to Y$ between metric spaces is called an *L-Lipschitz* mapping, if

$$d_Y(f(x), f(y)) \leq L d_X(x,y)$$

for every pair of points $x, y \in X$. We will have several situations where our mappings are not surjective, but merely such that all the points have a global upper bound for the distance to the image of our mapping. For this reason we do not demand in the following definition a Bi-Lipschitz map to be necessarily surjective.

A mapping $f\colon X \to Y$ between metric spaces is called an *L-bi-Lipschitz*, if for every pair of points $x, y \in X$ we have that

$$\frac{1}{L} d_X(x,y) \leq d_Y(f(x), f(y)) \leq L d_X(x,y).$$

A surjective bi-Lipschitz map is always a homeomorphism.

We will later need to use some basic results concerning topological dimension. We give the definition here, for further results and the proofs of the following results we refer to [HW41].

Definition 2.1. We define the topological dimension $\dim(X)$ of a topological space X in the following inductive way.

The empty set has dimension -1. A topological space X has *dimension at most* n, denoted $\dim(X) \leq n$, if every point of X has a neighbourhood basis consisting of sets whose boundaries have dimension at most $n-1$. The space X has *dimension exactly* n, if $\dim(X) \leq n$, and it is not true that $\dim(X) \leq n-1$.

Theorem 2.2. *If $f\colon X \to Y$ is a Lipschitz mapping between two metric spaces and $\dim(X) \leq n$, then $\dim(\mathrm{Im}(f)) \leq n$.*

2.1 Manifolds

We now define the basic element of study in this thesis; a manifold. We prove some useful topological properties of manifolds that we will use frequently throughout the rest of this thesis. Later in section 4.1 we will also define a path-metric structure for a manifold and prove some more sophisticated results. For a precise exposition of manifolds and their properties we refer to [Lee03].

Definition 2.3. An n-dimensional manifold $\mathcal{M}$ is a nonempty topological space with the following three properties:

($\mathcal{M}$1) Every point has a neighbourhood homeomorphic to an open subset of $\mathbb{R}^n$. Such neighbourhoods are called *chart-neighbourhoods*.

($\mathcal{M}$2) The topology of $\mathcal{M}$ has a countable basis. This property will be referred as the N_2-property.

($\mathcal{M}$3) The topology of $\mathcal{M}$ has the Hausdorff property.

Remark 2.4. The property ($\mathcal{M}$1) can be replaced with the following property that leads to an equivalent definition;

($\mathcal{M}$1') Every point has a neighbourhood homeomorphic to $\mathbb{R}^n$.

We shall use both of these equivalent properties in the sense that we can choose chart-neighbourhoods to be homeomorphic to the whole of $\mathbb{R}^n$ of just to some subdomain without special notice.

We will often use the following useful lemma when constructing countable bases with needed properties for a given topology.

Lemma 2.5. *Let X be a topological space with a countable basis for its topology. Let $\{\{\mathcal{V}^x\} \mid x \in X, i \in I\}$ be a collection of neighbourhood bases for all points of X. Then the topology of X has a countable basis consisting of sets $V_i^x \in \mathcal{V}^x$.*

Proof. We begin by noting that any topological space X with the N_2-property has also the so called *Lindelöf property* which states that every open cover of X has a countable subcover. This follows quite easily, because if we have an arbitrary open cover $\mathcal{D}$ of X, we may choose for each $x \in X$ a neighbourhood U from the countable basis that is a subset of some $A \in \mathcal{D}$. When we pick for each such neighbourhood U one element A of the cover $\mathcal{D}$ containing this neighbourhood we have the needed countable cover, as there can be at most countably many selections since the basis of the topology was assumed countable.

Now we show that if a topological space has a countable basis, then every basis of its topology has a countable subbasis. This will prove the claim as the collection $\{\mathcal{V}_i^x \mid x \in X, i \in I\}$ forms a basis of our topology. Let $\mathcal{A}$ be any basis for the topology of X, and denote by $\mathcal{U} = \{U_1, U_2, \ldots\}$ a countable basis of the topology of X. We first define collections $\mathcal{B}_i = \{B \in \mathcal{A} \mid B \subset U_i\}$. Each $\mathcal{B}_i$ is an open cover of the set U_i. Because U_i is a subset of a topological space with a countable basis it also has a countable basis. Especially U_i has the Lindelöf property, so there exists a countable subcover $\mathcal{B}_i^*$ of $\mathcal{B}_i$ covering U_i. Now the collection $\cup_{i \in \mathbb{N}} \mathcal{B}_i^*$ is a countable subbasis of $\mathcal{A}$. □

Lemma 2.6. *Every manifold is locally compact, i.e. every point has a neighbourhood with compact closure.*

Proof. Let $\mathcal{M}$ be a manifold and $x_0 \in \mathcal{M}$. We can pick a chart-neighbourhood (U, φ) of x_0 such that

$$\varphi\colon (U, x_0) \to (\mathrm{B}\left(0, 1\right), 0)$$

is a homeomorphism. Now the set $V := \varphi^{-1}[\mathrm{B}\left(0, 1/2\right)]$ is a neighbourhood of x_0 as $\varphi|_V\colon V \to \mathrm{B}\left(0, 1/2\right)$ is a homeomorphism. Furthermore,

$$\overline{V} = \overline{\varphi^{-1}[\mathrm{B}\left(0, 1/2\right)]} = \varphi^{-1}[\overline{\mathrm{B}\left(0, 1/2\right)}] = \varphi^{-1}[\overline{\mathrm{B}}(0, 1/2)],$$

so the closure of V is compact as all closed balls of $\mathbb{R}^n$ are compact and

$$\varphi|_{\overline{V}}\colon \overline{V} \to \overline{\mathrm{B}}(0, 1/2)$$

is a homeomorphism. □

We say that a topological space X is *σ-compact* if there exists a sequence $K_1, K_2, \ldots$ of compact subsets of X such that $\bigcup_{j\in\mathbb{N}} K_j = X$.

Corollary 2.7. *All manifolds are σ-compact.*

Proof. This follows by combining lemmas 2.5 and 2.6. □

A topological space is said to be *paracompact* if every open cover admits a locally finite refinement.

Lemma 2.8. *Every manifold is paracompact.*

Proof. We have already shown that all manifolds have the Lindelöf property. Manifolds are also regular as we now show. Let A be a closed subset of our manifold $\mathcal{M}$ and let $x \in \mathcal{M}\backslash A$. Let (U, φ) be a precompact chart-neighbourhood of x and pick a 'sub-chart neighbourhood' $V \subset U$ with $\overline{V} \subset U$. Now we find disjoint open neighbourhoods W_x and W'_A for x and $\overline{V} \cap A$, respectively, by using the fact that $\mathbb{R}^n$ is regular. Expanding W'_A to $W_A = W'_A \cup \complement\overline{V}$ we have the needed neighbourhoods of x and A.

Now a regular Lindelöf space is always paracompact, see for example [Wil04]. □

Remark 2.9. In some sources the topological N_2 property of topology is replaced by paracompactness in the definition of a manifold. These conditions yield equivalent definitions for connected manifolds.

2.2 The fundamental group of a manifold

We now define the fundamental group of a topological space and show that the fundamental group of a manifold cannot be too large. The fundamental group will serve us as a fundamental building block of our basic concepts.

Let X and Y be topological spaces, and let $f, g\colon X \to Y$ be continuous. The mappings f and g are said to be *homotopic* if there exists a mapping $F\colon [0, 1] \times X \to Y$ such that F is continuous, $F(0, x) = f(x)$ and $F(1, x) = g(x)$ for all $x \in X$. Such map F is called a *homotopy* from f to g. If we choose $A \subset X$ and $B \subset Y$, we may define the homotopy of pairs by demanding that for each $t \in [0, 1]$ we have a map $(x \mapsto F(x, t))\colon (X, A) \to (Y, B)$.

Let (X, x_0) be a topological space with a fixed point. The fundamental group $\Pi_1(X, x_0)$ of this pair is the set of homotopy-equivalence classes of loops $\gamma\colon ([0,1], \{0,1\}) \to (X, x_0)$. It is given a group structure from the composition of loops by setting $[\alpha][\beta] = [\alpha\beta]$ for all $[\alpha], [\beta] \in \Pi_1(X, x_0)$ and this group structure is well defined (see for example [Hat02, Propositions 1.2. and 1.3., p. 26]). If X is path connected and x, y are any two points in X, we can define an isomorphism ψ between the groups $\Pi_1(X, x)$ and $\Pi_1(X, y)$ by picking a path $\gamma\colon x \curvearrowright y$ and setting $\psi([\alpha]) = [\overleftarrow{\gamma}\alpha\gamma]$. Thus we can omit the fixed point and just write $\Pi_1(X)$ when the specific fixed point is not of interest.

We say that two topological spaces X and Y are *homotopy-equivalent* if there exists continuous mappings $f\colon X \to Y$ and $g\colon Y \to X$ and homotopies $F_X\colon X \times [0,1] \to X$ and $F_Y\colon Y \times [0,1] \to Y$ from $g^{-1} \circ f$ to id_X and from $f^{-1} \circ g$ to id_Y, respectively. Either of the mappings f or g is called a homotopy-equivalence. A topological space is said to be *simply connected* if its fundamental group is trivial, i.e. it consists of a single element. A topological space X is called *contractible* if there exists a point $x_0 \in X$ such that the constant mapping $\iota\colon X \to \{x_0\}$ is a homotopy-equivalent to the identity mapping id_X.

Lemma 2.10. *A contractible topological space is always path-connected and simply connected.*

Proof. Let $F\colon X \times [0,1] \to X$ be a homotopy between constant mapping $\iota\colon X \to \{x_0\}$ and the identity mapping.

Let $y \in X$. We note that the path $\gamma\colon [0,1] \to X$ defined by $\gamma(t) = F(y,t)$ connects y to x_0. With this we can connect any two points in X.

For any loop γ we note that $G\colon [0,1] \times [0,1] \to X$ defined by $G(s,y) = F(\gamma(t), s)$ is a homotopy that shrinks γ to a point, and thus X is simply connected. $\square$

We will need the following result later:

Theorem 2.11. *The fundamental group of a manifold is always countable.*

The following proof is from [Lee03, Theorem 8.11., p. 189] with some added explanations and details. The basic idea is represented in figure 1.

Proof. We know by lemma 2.5 that a manifold has a countable cover consisting of sets homeomorphic to the euclidean unit ball. Let us denote this cover by $\mathcal{B}$. We see that for any two $B, B' \in \mathcal{B}$ the set $B \cap B'$ is homeomorphic to an open subset of $\mathbb{R}^n$ and thus has at most countably many components. (Note that as open connected subsets of $\mathbb{R}^n$ these components are also path-connected.) We denote by $\mathcal{X}$ the set in which we pick a point from each of the components of the intersections $B \cap B'$, where $B, B' \in \mathcal{B}$. We emphasize that the set $\mathcal{X}$ is countable. For every pair of points $x_i, x_j \in \mathcal{X} \cap B$ for some $B \in \mathcal{B}$ we pick a path $\alpha_{i,j}\colon [0,1] \to B$ connecting the points x_i and x_j. We call these paths *special paths*. There is clearly only a countable number of special paths, so there is only a countable amount of finite compositions of special paths.

Let us now take an arbitrary path $\gamma\colon [0,1] \to \mathcal{M}$. As our result is for the fundamental group, we shall eventually be interested in loops with a fixed base point. Because the change of base point does not affect the algebraic structure of the fundamental group, we may assume for our result that the endpoints of γ lie in $\mathcal{X}$. We want to show that γ is homotopic to a finite composition of special

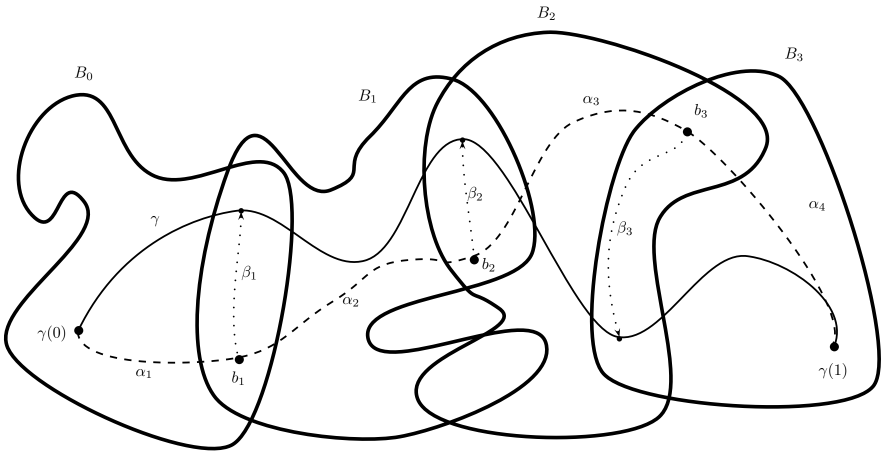

Figure 1: Replacing the path γ with a finite composition of special paths.

paths. The pre-images of the open sets in $\mathcal{B}$ give an open cover for the compact set $[0,1]$, so by choosing a Lebesgue number for this cover we acquire a division

$$0 = a_0 < a_1 < \cdots < a_n = 1$$

of the interval $[0,1]$ such that $\gamma\,[a_i, a_{i+1}] \subset B_i$ for some $B_i \in \mathcal{B}$ for all $i = 0, \ldots, n-1$. We abbreviate $\gamma_i := \gamma|_{[a_i, a_{i+1}]}$, and we denote a fixed set $B \in \mathcal{B}$ with $\gamma([a_i, a_{i+1}]) \subset B$ by B_i. For $0 < i < n$ the point $\gamma_i(a_i)$ lies is some component of $B_{i-1} \cap B_i$. Denote by b_i the element of $\mathcal{X}$ in this component and by β_i the path connecting b_i and $\gamma_i(a_i)$ within the component. We also set β_0 and β_{n+1} to be the constant paths concerning the points $\gamma(0)$ and $\gamma(1)$, respectively. Finally, let us denote by α_i the special path connecting b_{i-1} and b_i. (So with certain enumeration of $\mathcal{X}$ we would have $\alpha_i = \alpha_{i-1,i}$ in the sense of previously used notation for special paths.)

Now we see that as the sets $B \in \mathcal{B}$ are simply connected as homeomorphic images of Euclidean balls, we have that each path γ_i is homotopic to the composition of paths $\overleftarrow{\beta}_i \alpha_i \beta_{i+1}$. Thus

$$\gamma = \gamma_0 \gamma_1 \cdots \gamma_{n-1} \sim \left(\overleftarrow{\beta}_0 \alpha_0 \beta_1\right)\left(\overleftarrow{\beta}_1 \alpha_1 \beta_2\right) \cdots \left(\overleftarrow{\beta}_n \alpha_n \beta_{n+1}\right) \sim \alpha_0 \alpha_1 \cdots \alpha_n$$

and the path γ is homotopic to a finite composition of special paths.

Now especially in every homopopy class of loops in $\mathcal{M}$ there is a representative that is a finite composition of special paths. Because there is only a countable number of such compositions, is the fundamental group necessarily also countable.

□

2.3 Covering space of a manifold

We next turn our attention to the concept of a (universal) cover of a topological space. This is a useful concept for we can in a sense 'open up' a topological space, or in our case usually a manifold. This enables us later on to talk about concepts of volume growth and isoperimetric inequalities that sometimes rely in their use to the possibility of increasing radius of balls indefinitely without 'running out of space.' So by opening up a manifold that is too 'small' into its universal cover we have better chances of ending up with an object in which large-scale operations can be implemented. The exposition of universal covers is loosely based on [Hat02]. All results that are said to follow from the basic results of algebraic topology can be found from this book.

Definition 2.12. Let X, Y be topological spaces. We call a continuous surjective mapping $f: Y \to X$ a *covering map* if for any point $x \in X$ there exists a neighbourhood U of x in X so that $p^{-1}[U]$ is a disjoint union of open sets V_i for which $p|_{V_i}: V_i \to U$ is a homeomorphism for any $i \in I$. We shall call such a neighbourhood U a *covering neighbourhood* (of the point x). It is quite clear that any point has a neighbourhood basis consisting of covering neighbourhoods.

Let X be a connected topological space. We say that a pair (Y, p) is a *covering space of* X if Y is a connected topological space and $p: Y \to X$ is a covering map.

We say that a covering space (Y_1, p_1) of a topological space X is *universal*, if for any other covering space (Y_2, p_2) of X we have a covering map $f: Y_1 \to Y_2$ such that (Y_1, f) is a covering space of Y_2 and $p_1 = p_2 \circ f$. By basic results of homotopy theory a cover is universal if and only if it is simply connected and the universal cover of a topological space, when it exists, is essentially unique. We shall not prove the following basic theorem concerning the existence of universal covering spaces; the proof can be found e.g. in [Hat02, p. 64]. To state the result we define a topological space X to be *semilocally simply connected* if every point $x \in X$ has a neighbourhood U such that the mapping $f: \Pi_1(U, x) \to \Pi_1(X, x)$ induced by the inclusion $U \hookrightarrow X$ is trivial.

$$\begin{tikzcd}
Y_1 \arrow[dr, "f"] \arrow[dd, "p_1"'] & \\
& Y_2 \arrow[dl, "p_2"] \\
X &
\end{tikzcd}$$

Figure 2: Ordering covers

Lemma 2.13. *Let X be a topological space that is* path connected*,* locally path connected *and* semilocally simply connected*.*

Then X has a universal cover.

We shall show that every manifold has an universal cover. We do, however, separate part of the proof as a lemma since we need it later in a slightly different context.

Lemma 2.14. *Let $\mathcal{M}$ be a connected topological space that is locally path connected. Then $\mathcal{M}$ is path connected.*

Proof. We shall call the path connected open sets of $\mathcal{M}$ *chart-neighbourhoods* as they imitate chart-neighbourhoods of manifolds. We say that two points x_1 and x_2 are *chart-connected* if there exists a finite sequence $U_1, \dots, U_k$ of chart-neighbourhoods of $\mathcal{M}$ such that $U_i \cap U_{i+1} \neq \emptyset$ for all i, and $x_1 \in U_1$, $x_2 \in U_k$.

Let $x_0 \in \mathcal{M}$. We wish to show that the set

$$A = \{x \in \mathcal{M} \mid \text{The point } x \text{ is chart-connected to the point } x_0\}$$

is the whole of $\mathcal{M}$ because elements within A are easy to connect with paths lying in in A. We do this by showing that the set A is both open and closed because a nonempty (clearly $x_0 \in A$) subset of a connected space that is both open and closed has to be the whole space.

Assume $x \in \partial A$ and let V be a chart neighbourhood of x. As $V \cap A \neq \emptyset$, we can pick a point $y \in V \cap A$ and connect x_0 to y with charts $U_1, \ldots, U_k$. Now the sequence $U_1, \ldots, U_k, V$ is a chart-connection of x_0 to x. On the other hand also $V \cap \complement A \neq \emptyset$, so we can pick $z \in V \cap \complement A$. But not the sequence $U_1, \ldots, U_k, V$ connects x_0 to z, so $z \in A$. This is a contradiction, so we must have $\partial A = \emptyset$. Thus A is both open and closed.

This means, as mentioned, that $\mathcal{M} = A$. Moreover, any two points $a, b \in \mathcal{M}$ can be chart-connected by first connecting a to x_0 and then x_0 to b because chart-connectivity is a transitive relation.

Now let x and y be arbitrary points in $\mathcal{M}$. Let us take a finite chain of chart neighbourhoods $U_1, \ldots, U_k$ connecting these two points and pick points $x_i \in U_i \cap U_{i+1}$. As the chart-neighbourhoods were assumed path connected, we find paths connecting each x_i to x_{i+1} in U_{i+1} and two paths connecting x to x_1 and x_{k-1} to $x \in U_1$ and $y \in U_k$, respectively. Composing these paths gives us a path connecting x and y within the set $\bigcup_i U_i \subset \mathcal{M}$. $\square$

Theorem 2.15. *Every connected topological manifold has a universal cover.*

Proof. We shall use lemma 2.13. Let $\mathcal{M}$ be a connected manifold. To see that $\mathcal{M}$ is locally path connected let $x \in \mathcal{M}$. As $\mathcal{M}$ is a manifold, there exists a chart neighbourhood U of x homeomorphic to $\mathbb{R}^n$ via a homeomorphism $f \colon U \to \mathbb{R}^n$. Now if we pick any two points a and b from this neighbourhood, there exists a path α in $\mathbb{R}^n$ connecting the points $f(a)$ and $f(b)$. The path $f^{-1} \circ \alpha$ connects the points a and b in U. Thus the manifold $\mathcal{M}$ is locally path connected. Now $\mathcal{M}$ is path connected by lemma 2.14.

To show that $\mathcal{M}$ is semilocally simply connected[1] we again let $x \in \mathcal{M}$ and we pick a chart neighbourhood U of x homeomorphic to $\mathbb{R}^n$. The space $\mathbb{R}^n$ is simply connected, so $\Pi_1(U) \simeq \Pi_1(\mathbb{R}^n) = 0$. Thus the inclusion-induced mapping is trivially trivial and $\mathcal{M}$ is thus semilocally simply connected.

So by our lemma 2.13 any manifold has a universal cover. $\square$

Now we shall look at some of the basic lifting properties of covering spaces. We will not give the proofs to these fundamental results, but they can be found from any basic book concerning homotopy theory, for example from [Hat02]. These theorems are stated for general covers and topological spaces, but in practice we will use them mostly for the universal cover of a manifold. We denote by $(\widetilde{\mathcal{M}}, p_{\widetilde{\mathcal{M}}})$ the universal cover of a manifold $\mathcal{M}$ and reserve the notation $(\widehat{\mathcal{M}}, p_{\widehat{\mathcal{M}}})$ for general cover of $\mathcal{M}$.

Definition 2.16. Let f be a map $f \colon X \to \mathcal{N}$, where $\mathcal{N}$ is a connected topological space, and $(\widehat{\mathcal{N}}, p_{\widehat{\mathcal{N}}})$ its cover. A *lift* of f is a map $\widetilde{f} \colon X \to \widehat{\mathcal{N}}$ such that $p_{\widehat{\mathcal{N}}} \circ \widetilde{f} = f$. (See figure 2.3.)

The lift of a function f is denoted $\widetilde{f}$.

[1] We actually show a stronger result that says $\mathcal{M}$ is actually *locally* simply connected.

Example 2.17. Let us take a path $f\colon [0,1] \to S^1$, $f(t) = \mathrm{e}^{4\pi it}$. This path circles the unit circle twice, beginning from the point $(0,1)$. The universal cover of the circle S^1 is $\mathbb{R}$ with the mapping $p(x) = \mathrm{e}^{2\pi it}$ as a covering map. In this case the lifts for f would be maps $\widetilde{f}_n\colon [0,1] \to \mathbb{R}$, $\widetilde{f}_n(t) = n + 2t$.

More generally maps of the form $f(t) = \mathrm{e}^{2\pi ig(t)}$, where $g\colon [0,1] \to \mathbb{R}$ is continuous, have lifts $\widetilde{f}_n\colon [0,1] \to \mathbb{R}$, $\widetilde{f}_n(t) = n + g(t)$.

Definition 2.18. Let $f\colon \mathcal{M} \to \mathcal{N}$ be a function between two topological spaces with covers $(\widehat{\mathcal{M}}, p_{\widehat{\mathcal{M}}})$ and $(\widehat{\mathcal{N}}, p_{\widehat{\mathcal{N}}})$. We say that a *double lift* of f is a map $\widetilde{\widetilde{f}}\colon \widehat{\mathcal{M}} \to \widehat{\mathcal{N}}$ such that $p_{\widehat{\mathcal{N}}} \circ \widetilde{\widetilde{f}} = f \circ p_{\widehat{\mathcal{M}}}$, i.e. it is a lift of the mapping $f \circ p_{\widehat{\mathcal{M}}}$. (See figure 2.3.)

The double lift of a function f is denoted $\widetilde{\widetilde{f}}$ and as it is also a lift, the following theorems for lifts work also for the double lift with the right interpretation.

$$\begin{tikzcd} & \widehat{\mathcal{N}} \arrow[d, "p_{\mathcal{N}}"] \\ X \arrow[ur, bend left, "\widetilde{f}"] \arrow[r, "f"] & \mathcal{N} \end{tikzcd} \qquad \begin{tikzcd} \widehat{\mathcal{M}} \arrow[r, "\widetilde{\widetilde{f}}"] \arrow[d, "p_{\mathcal{M}}"] & \widehat{\mathcal{N}} \arrow[d, "p_{\mathcal{N}}"] \\ \mathcal{M} \arrow[r, "f"] & \mathcal{N} \end{tikzcd}$$

Figure 3: A lift and a double lift of a function.

We only state the following basic theorem called the *lifting criterion*. The proof can be found at [Hat02, Proposition 1.33., p. 61].

Theorem 2.19. *Let $(\mathcal{M}, x_0)$ be a topological space, $\left((\widehat{\mathcal{M}}, \widehat{x}_0), p_{\widehat{\mathcal{M}}}\right)$ its covering space and $f\colon (Y, y_0) \to (\mathcal{M}, x_0)$ a continuous map from a path connected, locally path connected topological space Y. There exists a lift $\widetilde{f}$ of f if and only if*

$$f_* \left[\Pi_1 \left(Y, y_0\right)\right] \subset (p_{\mathcal{M}})_* \left[\Pi_1 \left(\widehat{\mathcal{M}}, \widehat{x}_0\right)\right].$$

This instantly gives us two corollaries, which shall be the only form of this theorem that we need. We omit the proofs as these results follow immediately from the lifting criterion.

Corollary 2.20. *Mappings from a simply connected, path connected and locally path connected domain always have a lift. Especially mappings from a simply connected manifold always have a lift to any cover.*

Corollary 2.21. *Let $\mathcal{M}$ be a manifold. Paths $\gamma\colon [0,1] \to \mathcal{M}$ always have a lift $\widetilde{\gamma}\colon [0,1] \to \widehat{\mathcal{M}}$.*

We will also need the following uniqueness of lifts. Proof can be found for example from [Hat02, Proposition 1.34., p.62].

Theorem 2.22. *If two lifts of a continuous function agree on one point of the domain and if the domain is connected, then the lifts agree on all of the domain.*

The following result is called the *the homotopy lifting property*, but we shall only need one of its corollaries later. Proof can be found for example from [Hat02, Proposition 1.30., p.60].

Theorem 2.23. *Suppose we have a covering space* $(\widehat{\mathcal{M}}, p_{\widehat{\mathcal{M}}})$ *of a topological space* $\mathcal{M}$*, a homotopy* $F\colon [0,1] \times Y \to \mathcal{M}$ *and a map* $\widetilde{f}_0\colon Y \to \widehat{\mathcal{M}}$ *lifting* f_0*. Then there exists a unique homotopy* $\widetilde{f}_t\colon Y \to \widehat{\mathcal{M}}$ *of* $\widetilde{F}$ *that lifts* F*.*

Corollary 2.24. *Two loops with the same base point in* $\mathcal{M}$ *are non-homotopic if their lifts with respect to some cover with the same starting point have different endpoints.*

Corollary 2.25. *Two loops with the same base point in* $\mathcal{M}$ *are homotopic if and only if their lifts to the universal cover with the same starting point have a common endpoint.*

We need next to construct a manifold structure on $\widehat{\mathcal{M}}$.

Theorem 2.26. *Let* $\mathcal{M}$ *be a manifold. Then the any of its covers is also a manifold.*

Proof. We need to show three things: every point in $\widehat{\mathcal{M}}$ has a neighbourhood homeomorphic to an open subset of $\mathbb{R}^n$, the topology of $\widehat{\mathcal{M}}$ has the Hausdorff property and the topology of $\widehat{\mathcal{M}}$ has a countable basis.

We first construct a neighbourhood homeomorphic to a domain of $\mathbb{R}^n$ for an arbitrary point in $\widehat{\mathcal{M}}$. The basic idea of the process is shown in picture 4. Let $x \in \widehat{\mathcal{M}}$. We note that $p(x)$ is a point in $\mathcal{M}$, so it has a neighbourhood U homeomorphic to an open subset of $\mathbb{R}^n$ via a map f. But by the definition of a covering space, the point $p(x)$ has also a covering neighbourhood V such that $p^{-1}[V]$ consists of disjoint open sets, each homeomorphic to V via the restriction of p. From these open sets we choose the unique one containing x and call it W. Now $V \cap U \subset \mathcal{M}$ is nonempty and homeomorphic to an open subset of $\mathbb{R}^n$ via the restriction of f to this intersection. On the other hand the set $V \cap U$ is homeomorphic to the set

$$\left((p|_W)^{-1}|_{V\cap U}\right)[V \cap U] \subset \widehat{\mathcal{M}},$$

which is a neighbourhood of x as an image of an open set under a homeomorphism. Thus the point x has a neighbourhood homeomorphic to an open subset of $\mathbb{R}^n$.

To show that the topology of the covering space has the Hausdorff property, let $x_1, x_2 \in \widehat{\mathcal{M}}$, $x_1 \neq x_2$. Let us study this in two cases.

Let us first assume that $p(x_1) \neq p(x_2)$. The points $p(x_1)$ and $p(x_2)$ lie in the manifold $\mathcal{M}$ which is a Hausdorff space. Thus there exists neighbourhoods U_1 and U_2 of these points such that $U_1 \cap U_2 = \emptyset$. We now claim that the neighbourhoods $p^{-1}[U_i]$ of x_1 and x_2 are disjoint. This follows quickly, as if there were a point

$$y \in p^{-1}U_1 \cap p^{-1}U_2,$$

then $p(y)$ would belong to both U_1 and U_2. This is impossible as these sets were chosen disjoint. Thus in the first case we find disjoint neighbourhoods for the points x_1 and x_2.

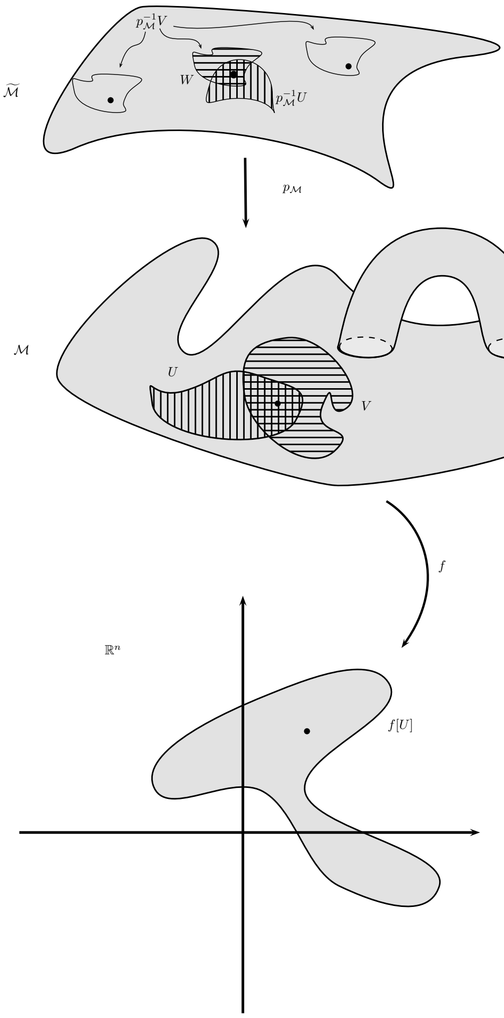

Figure 4: Constructing a chart neighbourhood in $\widehat{\mathcal{M}}$.

Let us now assume that $p(x_1) = p(x_2)$. By the definition of a covering space, the point $p(x_1)$ has a covering neighbourhood V such that $p^{-1}V$ is a disjoint collection of open sets V_i each homeomorphic to V. The points x_1 and x_2 cannot lie in a same component of $p^{-1}V$, because $p|_{V_i} \colon V_i \to V$ is as a homeomorphism especially an injection for all i. By picking the two unique open sets containing x_1 and x_2, we find the needed disjoint neighbourhoods.

Showing that the topology of a cover of a manifold has a countable base is a bit nontrivial. To appreciate this aspect one may spend time thinking why the Alexandroff long line is not a covering space for $\mathbb{R}$ or S^1. What we shall first show is that for any[2] $x_0 \in \mathcal{M}$ the set $p^{-1}\{x_0\}$ is countable.

Assume the opposite; there exists $x_0 \in \mathcal{M}$ such that the pre-image of this

[2]Actually it is a basic result of covering spaces that the sets $p^{-1}\{x\}$ have the same cardinality for all $x \in \mathcal{M}$, so it would suffice to show the result for just one fixed point. The proof would be essentially the same.

point under the covering map p is uncountable. The cover $\widehat{\mathcal{M}}$ is locally homeomorphic to $\mathbb{R}^n$ so it is especially locally path connected. This means that $\widehat{\mathcal{M}}$ meets the assumptions of lemma 2.14 and is path connected. So if we pick one point, say $y_0 \in p^{-1}\{x_0\}$, we may now construct paths γ_α connecting y_0 to each element of the set $p^{-1}\{x_0\}$. These paths give us uncountably many loops $p \circ \gamma_\alpha$ in $\mathcal{M}$, all with the same starting point x_0 and mutually non-homotopic by application of corollary 2.24. This, on the other hand is in contradiction with theorem 2.11. Thus we must have that $p^{-1}\{x_0\}$ is countable.

We now wish to construct a countable basis for the topology of $\widehat{\mathcal{M}}$. Note, that for every point $x \in \mathcal{M}$ the covering neighbourhoods V form a neighbourhood basis of x so by theorem 2.5 there exists a countable basis for the topology of $\mathcal{M}$ consisting of covering neighbourhoods. Call this basis $\mathcal{A}$. We set

$$\mathcal{B} = \left\{ U \subset \widehat{\mathcal{M}} \mid U \text{ is a component of } p^{-1}V \text{ for some } V \in \mathcal{A} \right\}$$

and show that $\mathcal{B}$ is the wanted basis. The set $\mathcal{B}$ is countable by our previous argument.

Let $W \subset \widehat{\mathcal{M}}$ be a connected neighbourhood of a point $y \in \widehat{\mathcal{M}}$. It suffices to find $B \in \mathcal{B}$ such that $y \in B \subset W$ to prove the claim. We note that by definition of a covering space there exists a neighbourhood V of $p(y)$ in $\mathcal{M}$ such that there is a component V_0 of $p^{-1}[V]$ containing y such that $p|_{V_0} \colon V_0 \to V$ is a homeomorphism. The restriction $p|_{W \cap V_0} \colon W \cap V_0 \to p\,[W \cap V_0]$ is still a homeomorphism and the set $p[W \cap V_0]$ a covering neighbourhood. Now $p\,[W \cap V_0]$ contains an element $A \in \mathcal{A}$ with $p(y) \in A$. Calling this component B, we see that $B \in \mathcal{B}$ and $y \in B \subset W$. This proves the claim. $\square$

3 Growth rate

We now begin to define the growth rate of a metric space. This will be our invariant with which we will be able to deduce non-existence of BLD mapping between manifolds in certain cases. We begin by looking at coarse quasi-isometries which are the moral isomorphisms in the category of coarse geometry.

3.1 Coarse quasi-isometries

We begin by studying coarse quasi-isometries which are the fundamental tool of coarse methods. Coarse quasi-isometries are a quite natural generalization of isometries and have a very concrete definition. However the small change in the definition gives us much more flexibility and makes the coarse quasi-isometry an essential tool in the studies of growth rate and other co-local properties of metric spaces. Two of the most important properties for coarse quasi-isometries for us are that firstly, they will actually preserve growth rate as well as it can be reasonably preserved and secondly, their existence gives an equivalence relation between metric spaces.

A subset A of a metric space X is said to be *full* in X if there exists a constant $\varepsilon \geq 0$ such that

$$\mathrm{B}\left(A, \varepsilon\right) := \{x \in X \mid \inf\{d(x,a) \mid a \in A\} \leq \varepsilon\} = X.$$

If we want to emphasize the constant ε, we can use the term *ε-full*.

Definition 3.1. Let (X, d_X) and (Y, d_Y) be metric spaces. We call a mapping $f\colon X \to Y$ a *coarse quasi-isometry* if the following two conditions hold.

(Q1) There exists a constants $C > 0$ and $D \geq 0$ such that

$$C^{-1} d_X(x,y) - D \leq d_Y(f(x), f(y)) \leq C d_X(x,y) + D$$

holds for all pairs of points in X.

(Q2) The image of f is full in Y.

Two spaces are called *coarsely quasi-isometric* if there exists a coarse quasi-isometry between them.

Note that a coarse quasi-isometry need not be continuous. An intuitive approach would be to think that two spaces are coarsely quasi-isometric if they look similar 'when looked from far away' or on large scales. The relation between metric spaces of being coarsely quasi-isometric is however an equivalence relation. Especially if $f\colon X \to Y$ is a coarse quasi-isometry such that $\mathrm{Im}(f)$ is ε-full in Y, we can define $g\colon Y \to X$ by picking for each for each $y \in Y$ a point $x \in X$ such that $d(y, f(x)) \leq \varepsilon$, and setting $g(y) = x$. We call this mapping g a *coarse inverse* of f and shall often abuse notation and denote in this situation $f^{-1} := g$, even though it is rare that we should have $f \circ f^{-1} = \mathrm{id}_Y$ or $f^{-1} \circ f = \mathrm{id}_X$, and the coarse inverse f^{-1} is unique only in the rarest of cases.

The following lemmas will be used throughout the rest of this thesis.

Lemma 3.2. *Let A be a subset of a metric space X. The inclusion mapping $A \hookrightarrow X$ is a coarse quasi-isometry if and only if the set A is full in X.*

Proof. Requirement ($Q1$) follows as in the induced metric the inclusion is an isometry. The image is full exactly when the set A is, so the inclusion is a coarse quasi-isometry exactly in this case. □

Lemma 3.3. *Let X, X', Y and Y' be metric spaces with $f\colon X \to X'$ and $g\colon Y \to Y'$ as coarse quasi-isometries. Then the spaces $X \times Y$ and $X' \times Y'$ are coarsely quasi-isometric when equipped with the product metric defined by*

$$d_{A\times B}\left((x, y), (z, w)\right) := d_A(x, z) + d_B(y, w).$$

Proof. The claim follows immediately by writing out the conditions for coarse quasi-isometry for the mapping

$$h := f \times g, \quad h(x, y) = (f(x), g(y)).$$

Let $(x, y), (a, b) \in X \times Y$. We note that

$$\begin{aligned} d_{X'\times Y'}\left(h(x, y), h(a, b)\right) &= d_{X'\times Y'}\left((f(x), g(y)), (f(a), g(b))\right) \\ &= d_{X'}(f(x), f(a)) + d_{Y'}(g(y), g(b)) \\ &\leq C_f d_X(x, a) + D_f + C_g d_Y(y, b) + D_g \\ &\leq C d_{X\times Y}\left((x, y), (a, b)\right) + D, \end{aligned}$$

where $C = \max(C_f, C_g)$ and $D = D_f + D_g$.

In a similar fashion we can see that

$$d_{X'\times Y'}\left(h(x, y), h(a, b)\right) \geq C^{-1} d_{X\times Y}\left((x, y), (a, b)\right) - D,$$

where again $C = \max(C_f, C_g)$ and $D = D_f + D_g$.

Now let us assume that $\operatorname{Im}(f)$ is ε_f-full in Y and $\operatorname{Im}(g)$ is ε_g-full in Y'. We note that $\operatorname{Im}(h) = \operatorname{Im}(f) \times \operatorname{Im}(g)$, so if we take any point $(x, y) \in X' \times Y'$, we see that

$$d_{X'\times Y'}((x, y), \operatorname{Im}(h)) = d_{X'}(x, \operatorname{Im}(f)) + d_{Y'}(y, \operatorname{Im}(g)) \leq \varepsilon_f + \varepsilon_g.$$

Thus the image of h is $(\varepsilon_f + \varepsilon_g)$-full in $X' \times Y'$.

□

Example 3.4.

1. The inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$ is a coarse quasi-isometry by lemma 3.2.
2. The inclusion $\mathbb{Q}^2 \hookrightarrow \mathbb{R}^2$ is a coarse quasi-isometry by lemma 3.2.
3. Any two bounded non-empty metric spaces are coarsely quasi-isometric. This can be seen by taking any constant map $f\colon X \to Y$ and picking the additive constants large enough:

$$d(x, y) - d(X) \leq d(f(x), f(y)) \leq d(x, y).$$

   This satisfies the (Q1)-condition because $d(f(x), f(y))$ is always zero, and for all $x, y \in X$ we have that $d(x, y) \in [0, d(X)]$. The condition ($Q2$) is satisfied as the set $f[X]$ is $d(Y)$-full in Y.

4. If X and Y are coarsely quasi-isometric, then $X = \emptyset$ if and only if $Y = \emptyset$. This can be seen by the fact that the image of an empty set can be ε-full only in an empty set, and that there exists no mappings from a nonempty set to the empty set.

5. The inclusion $\mathbb{N} \hookrightarrow \mathbb{R}$ is not a coarse quasi-isometry because the image of the set $\mathbb{N}$ is not ε-dense in $\mathbb{R}$. This can be seen by noting that for any $\varepsilon \in \mathbb{R}_+$ we have that $-\varepsilon - 1 \notin \mathrm{B}\left(\mathbb{N}, \varepsilon\right)$.

6. The mapping $\mathbb{N} \to \mathbb{R}$, $n \mapsto (-1)^n n$ is not a coarse quasi-isometry because the distance between the images $(-1)^n n$ and $(-1)^{n+1}(n+1)$ of two consecutive points is not bounded from above when n grows. The image is full however as it consists of all the positive odd integers and all negative even integers, so the image is 2-full.

   Actually the metric spaces $\mathbb{N}$ and $\mathbb{R}$ are not coarsely quasi-isometric. We will prove this in remark 3.41.

7. Let X and Y be nonempty metric spaces with Y is bounded. For any fixed $y_0 \in Y$ the natural injection.

$$\iota\colon X \to X \times Y, \quad \iota(x) = (x, y_0)$$

   is a coarse quasi-isometry

   This can be seen quite easily, as the mapping ι is an isometry, and we have that $\mathrm{B}\left(\mathrm{Im}(\iota), d(Y) + 1\right) = X \times Y$.

3.2 Growth rate of a metric space

We next turn to the concept of growth of a metric space. In the continuous setting we have no natural concept of volume on a manifold like we do in a Riemannian case. We work around this constraint in essence by approximating our manifolds with sufficiently discrete subsets called nets. For these subsets the growth rate, which represents in some sense a discretization of the Riemannian volume, can be calculated from the amount of points in a ball around a fixed point when the radius is increased. We will also give conditions for metric spaces under which we do not need to concern ourselves about the specific choice of an approximating net in any essential manner.

First we need to define the set of all possible growth rates. Let

$$F = \{f \colon \mathbb{R}_+ \to \mathbb{N} \cup \{\infty\} \mid f \text{ is monotone non-decreasing }\}$$

and define a reflexive transitive relation to this set by setting for $f, g \in F$ that $f \leq g$, if there exists non-negative constants C, D and r_0 such that

$$f(r) \leq Cg(Dr)$$

whenever $r \geq r_0$. We define an equivalence relation to this set by setting $f \sim g$ if $f \leq g$ and $g \leq f$, or equivalently, if there exists constants C, D and r_0 such that

$$C^{-1}g(D^{-1}r) \leq f(r) \leq Cg(Dr)$$

whenever $r \geq r_0$. The equivalence classes of of mappings under this relation are called growth rates and we denote the equivalence class of a function $f \in F$ by $\mathcal{O}(f)$. Finally we denote by $\mathcal{GR}$ the set of equivalence classes

$$F/\sim = \{\mathcal{O}(F) \mid f \in F\}.$$

We now show that the relation $\leq$ in F induces a well defined ordering to $\mathcal{GR}$. The proof is quite straightforward as we defined this quotient such that elements within an equivalence classes cannot be distinguished by the ordering.

Lemma 3.5. *The relation $\leq$ in F induces a well defined (partial) ordering to $\mathcal{GR}$.*

Proof. Let $\mathcal{O}(f), \mathcal{O}(g) \in \mathcal{GR}$ and $f' \in \mathcal{O}(f)$, $g' \in \mathcal{O}(g)$. We wish to show that $f \leq g$ if and only if $f' \leq g'$. Assuming $f \leq g$, there exists constants C_1, D_1 and r_0^1 such that $f(r) \leq C_1 g(D_1 r)$ for all $r \geq r_0^1$. But because $g \sim g'$, we have constants C_2, D_2 and r_0^2 such that $g(r) \leq C_2 g'(D_2 r)$ for all $r \geq r_0^2$. This implies that

$$C_1 g(D_1 r) \leq C_1 C_2 g'(D_1 D_2 r)$$

for all $r \geq \frac{r_0^2}{D_2}$. As we assumed that $f \sim f'$, we especially have constants C_3, D_3 and r_0^3 such that $f'(r) \leq C_3 f(D_3 r)$ for all $r \geq r_0^3$.

Combining all of the above and denoting $C = C_1 C_2 C_3$, $D_3 = D_1 D_2 D_3$ and $r_0 = \max\left\{r_0^1, \frac{r_0^2}{D_2}, \frac{r_0^3}{D_3}\right\}$ we have that $f'(r) \leq C g'(D)$ for all $r \geq r_0$. By the symmetry of the argument we see that the ordering relation is well defined.

Now as the original relation $\leq$ in F was reflexive and transitive, so is the induced relation in $\mathcal{GR}$. Furthermore the definition of the elements of $\mathcal{GR}$ guarantees that if $\mathcal{O}(f) \leq \mathcal{O}(g)$ and $\mathcal{O}(g) \leq \mathcal{O}(f)$, then $\mathcal{O}(f) = \mathcal{O}(g)$. Thus we have a natural ordering in $\mathcal{GR}$.

$\square$

The main theorems of this thesis will give invariants based to comparing growth rates of certain manifolds. We now give some vocabulary on certain usual types of growth rates. The growth rate of a function f (or of a metric space) is said to be *polynomial*, if there exists $n \in \mathbb{N}$ such that

$$\mathcal{O}(f) \leq \mathcal{O}(x \mapsto x^n) =: \mathcal{O}(x^n).$$

Growth rate of a polynomial order of 1 is called *linear* growth. If a growth rate is not of polynomial order, we say that it is of *superpolynomial order*. If $\mathcal{O}(f) = \mathcal{O}(x \mapsto \mathrm{e}^x)$, we say that the growth rate is of *exponential order* and if the growth greater than exponential order we say that it is of *super-exponential order*.

3.2.1 Growth rate of nets

We will define the growth rate of an arbitrary metric space by approximating the metric space in question with a sufficiently discrete subset defined as follows. A metric space X is *ε-separated*, if for all $x, y \in X$ we have either $x = y$ or $d(x, y) > \varepsilon$. Recall that a subset A of a metric space X is called ε-full, if $B(A, \varepsilon) = X$. (If we do not need to fix the constant ε we will just call sets *full* if the ε-fullness criterion is satisfied for some $\varepsilon \geq 0$.)

Definition 3.6. *An ε-net* of a metric space X is an ε-separated 2ε-full subset of X. If we call some set 'just' an ε-net without specifying a superset, the set is assumed to be an ε-net of itself, that is, an ε-separated metric space.

Nets can always be found as the following lemma shows.

Lemma 3.7. *There exists an ε-full ε-net in any metric space X.*

Proof. Let X be a metric space and let $\varepsilon > 0$. We define $\mathcal{A}$ to be the collection of all ε-separated subsets of X. We define a partial order in $\mathcal{A}$ by setting $A < B$ if $A \subset B$.

Let us assume that $\mathcal{B}$ is a linearly ordered subset of $\mathcal{A}$. We claim that $B_0 = \cup\mathcal{B}$ is an upper bound for $\mathcal{B}$. We have $B_0 > A$ for all $A \in \mathcal{B}$ as for all $A \in \mathcal{B}$ we clearly have that $A \subset \cup\mathcal{B} = B_0$. Thus we only need to check that $d(x, y) \geq \varepsilon$ for all disjoint pairs of points $x, y \in B_0$. But if $x, y \in B_0 = \cup\mathcal{B}$ are disjoint, there must exist sets $A_x, A_y \in \mathcal{B}$ containing the points x and y, respectively. The union defining B_0 is over a monotone collection of sets, so there must exist a set $A_{xy} \in \mathcal{B}$ containing both these points, and thus $d(x, y) \geq \varepsilon$.

Now by Zorn's lemma there exists a maximal element N of the collection $\mathcal{A}$. we next show that this is an ε-net of X. We need only to check that $\mathrm{B}\big(N, \varepsilon\big) = X$. But this is easy as if we had a point $x \in X$ with $d(x, N) > \varepsilon$, we could add this point to our net N. But that would be a contradiction with the maximality of N, so the claim holds. □

For ε-nets the growth rate can easily be defined in a unique way.

Definition 3.8. The *growth function* of an ε-net S with fixed point $x_0 \in S$ is defined to be the mapping

$$\Gamma_S^{x_0} \colon \mathbb{R}_+ \to \mathbb{N} \cup \{\infty\}, \quad \Gamma_S^{x_0}(r) = \sharp\, \mathrm{B}_S\big(x_0, r\big).$$

The growth rate of an ε-net (S, x_0) is the equivalence class

$$\mathrm{Ord}(S, x_0) := \mathcal{O}\left(\Gamma_S^{x_0}\right) \in \mathcal{GR}.$$

Note that any bi-Lipschitz change of metric does not affect the asymptotic behaviour of the growth function of a net.

The following theorem tells us that we can actually omit the fixed point and talk just about the growth of the net.

Theorem 3.9. *The growth rate of a net does not depend on the chosen base point, that is,* $\mathrm{Ord}(S, x_0) = \mathrm{Ord}(S, y_0)$ *for all* $x_0, y_0 \in S$*.*

Proof. Let $x_0, y_0 \in X$ be two base points. For any $n \in \mathbb{N}$

$$\mathrm{B}\big(x_0, n\big) \subset \mathrm{B}\big(y_0, n + d(x_0, y_0)\big) \quad \text{and} \quad \mathrm{B}\big(y_0, n\big) \subset \mathrm{B}\big(x_0, n + d(x_0, y_0)\big),$$

so we have for all $n \geq d(x_0, y_0)$ that

$$\mathrm{B}\big(x_0, n\big) \subset \mathrm{B}\big(y_0, n + d(x_0, y_0)\big) \subset \mathrm{B}\big(y_0, n + n\big) = \mathrm{B}\big(y_0, 2n\big).$$

Similarly,

$$\mathrm{B}\big(y_0, n\big) \subset \mathrm{B}\big(x_0, n + d(x_0, y_0)\big) \subset \mathrm{B}\big(x_0, n + n\big) = \mathrm{B}\big(x_0, 2n\big).$$

This means that we may actually choose $C = 1$, $D = 2$ and $n_0 = d(x_0, y_0)$ to get

$$\Gamma_X^{x_0}(\frac{n}{2}) \leq \Gamma_X^{y_0}(n) \leq \Gamma_X^{x_0}(2n)$$

for all $n \geq d(x_0, y_0)$ and thus the growth rate of a net does not depend on the chosen base point. $\square$

We denote by $\mathrm{Ord}(S)$ the equivalence class of the growth function of a net S. This notation is well defined by the previous theorem.

3.2.2 Growth class of a metric space

We now turn to the definition of a growth rate of an arbitrary metric space. We would *like* to define that the growth rate of a metric space X is the growth rate of any ε-net of X. This cannot be done because there exists metric spaces for which the choice of a different ε-net (even without changing the parameter ε) yield strictly different growth rate as we see in example 3.10.

Example 3.10. Let us denote by B the unit ball of the Banach space ℓ^∞. Denote by X_1 the space $B \times \mathbb{Z}$. We note that as a product of a metric space with a bounded metric space, the space X_1 is coarsely quasi-isometric to $\mathbb{Z}$. But if we look at any ε-net S with $\varepsilon \leq \frac{1}{4}$, we note that $\Gamma_S(r) = \infty$ for any $r \geq 1$. Thus X_1 is a metric space that looks like $\mathbb{Z}$ in the coarse sense, but has an infinite growth rate.

For a more sophisticated example, let $f \colon \mathbb{N} \to \mathbb{N}$ be any monotone non-decreasing function. Pick $\delta \in]0, \frac{1}{4}[$ We modify the space X_1 by changing each copy of the infinite-dimensional unit ball B at the point $k \in \mathbb{Z}$ to a cube B_k as follows. For $k < 0$ we take $B_k = \prod_{j \in \mathbb{N}} [-\delta, \delta]$. For $k \geq 0$ we take

$$B_k = \ell^\infty \cap \left(\left(\prod_{j=0}^{f(k+1)-f(k)} [-2\delta, 2\delta] \right) \times \left(\prod_{f(k+1)-f(k)}^{\infty} [-\delta, \delta] \right) \right).$$

call the result X_2. Now if we pick an ε-net v with $\varepsilon = 3\delta$, we see that $\mathrm{Ord}(V) = \mathcal{O}(f)$. So we have that this metric space is still coarsely quasi-isometric to $\mathbb{Z}$, but has the growth rate $[f]$.

Do note that if we pick an ε-net from either X_1 or X_2 with $\varepsilon \geq 2$ we always get a net with the growth rate of $\mathbb{Z}$. (This can be deduced directly, but this is also a corollary of theorem 3.19 in some sense.)

What happens in the previous example is a 'bad' thing. We would like our growth rate to mimic the Riemannian concept of volume and be preserved under coarse quasi-isometries. Because the volume growth of $\mathbb{R}$ is linear, we would like the metric space of integers to have linear growth rate as it is coarsely quasi-isometric to the metric space of real numbers. But in the example both of the metric spaces had nets with infinite or arbitrary growth rates even though the metric spaces themselves are coarsely quasi-isometric to $\mathbb{Z}$.

We will *need* the definition of growth rate to be such that it is preserved as strongly as possible under coarse quasi-isometries. Our hopes rise when we note that in the previous example we can get ε-nets to the metric space that resemble $\mathbb{Z}$ if we just make ε large enough. We should hope that under some criteria the same would always happen, meaning that we would always find the small 'real' growth rate of our metric space by taking nets sparse enough. In this we will succeed and the concrete result and condition for this to happen will be given definition 3.16 and theorem 3.18.

We continue now by defining the concept of growth rate for general metric spaces.

Definition 3.11. Let X be a metric space. Denote by $\mathcal{S}(X)$ the set of all nets in X. The growth class X is defined to be

$$\mathrm{Ord}_*(X) = \{\mathcal{O}(\Gamma_S) \mid S \in \mathcal{S}(X)\} \subset \mathcal{GR}.$$

Note that any bi-Lipschitz change of the metric does not affect $\mathrm{Ord}_*(X)$. The set $\mathrm{Ord}_*(X)$ can be large, even infinite. We would need in the coming results ways to find lower bounds to this set, the best possible situation being that where we have a minimal element in this set as seems to happen in the case of example 3.10. We will in theorem 3.18 show that a certain metric condition guarantees a minimal element in the growth class. Before that we prove a few auxiliary results about the ordering structure of $\mathrm{Ord}_*(X)$.

Lemma 3.12. *Let A be an ε-full subset of a metric space X. Then every ε-full ε-net S of A is an ε-net in X.*

Proof. By lemma 3.7 we can find nets with arbitrary net-constant ε from any metric space, and such that the resulting net is ε-full. Let us now take some ε-net from the set A with this property and call it V. To show that the set V is net in X we need to show that it is ε-full in X. Take $x \in X$. As the set A was assumed ε-full in X we have a point $a \in A$ such that $d(a, x) < \varepsilon$. Now as V was an ε-net in A we have $v \in V$ such that $d(v, a) < \varepsilon$. By triangle inequality we now have

$$d(x, v) \leq d(x, a) + d(a, v) < \varepsilon + \varepsilon = 2\varepsilon.$$

This proves the claim. □

Remark 3.13. Note that the previous theorem gives us ε-nets contained in full subsets with arbitrary large ε.

Theorem 3.14. *Let V be an ε-net in a metric space X. For any $k\varepsilon$-net P in X with $k \geq 4$ we have that* $\mathrm{Ord}(P) \leq \mathrm{Ord}(V)$*.*

Proof. Let V be an ε-net in X and P and $k\varepsilon$-net in X with $k \geq 4$. Fix points $v_0 \in V$ and $p_0 \in P$.

As V is an ε-net in X we have $d(p, V) < 2\varepsilon$ for all points $p \in P$. This means that for each $p \in P$ we can pick a point v from V with $d(p, v) < 2\varepsilon$. Denote by f the mapping $P \to V$ thus defined. We note that as P was assumed an $k\varepsilon$-net, we have for any disjoint x and y in V that $d(x, y) \geq k\varepsilon$. Also by definition of f we have that $d(f(x), x) < 2\varepsilon$ for all $x \in P$. Now by triangle inequality we have that for any $x, y \in P$, $x \neq y$

$$d(x, y) \leq d(x, f(x)) + d(f(x), f(y)) + d(f(y), y) < 2\varepsilon + 2\varepsilon + d(f(x), f(y)).$$

So

$$d(f(x), f(y)) > d(x, y) - 4\varepsilon \geq k\varepsilon - 4\varepsilon \geq 4\varepsilon - 4\varepsilon \geq 0.$$

Thus the mapping f is injective and it maps the the set $\mathrm{B}\left(x, r\right) \cap P$ into the set $\mathrm{B}\left(f(x), r + 2\varepsilon + d(v_0, p_0)\right) \cap V$ by similar calculation as above. Denoting $\mathrm{B}_A\left(x, r\right) := A \cap \mathrm{B}\left(x, r\right)$ we now have

$$\Gamma_P^{p_0}(r) = \sharp\, \mathrm{B}_P\left(p_0, r\right) \leq \sharp\, \mathrm{B}_V\left(v_0, r + 2\varepsilon + d(v_0, p_0)\right) = \Gamma_V^{v_0}(r + 2\varepsilon + d(v_0, p_0)).$$

Thus by choosing $r_0 = 2\varepsilon + d(v_0, p_0)$ we have $\Gamma_P^{p_0}(r) \leq \Gamma_V^{v_0}(2r)$ for all $r \geq r_0$ and this proves the claim. $\square$

Theorem 3.15. *Let P and S be nets in a metric space X. Then there exists a net V in X with* $\mathrm{Ord}(V) \leq \mathrm{Ord}(P)$ *and* $\mathrm{Ord}(V) \leq \mathrm{Ord}(S)$.

Proof. The claim follows from the previous theorem by picking any ε-net V in X with $\varepsilon \geq 4(\varepsilon_P + \varepsilon_S)$, where ε_P and ε_S are the net-constants of P and S, respectively. $\square$

Definition 3.16. A metric space X is called weakly doubling if the following condition is satisfied. For every pair of radii R and r there exists a constant $K := K(R, r)$ such that any ball with radius R can contain at most K disjoint balls with radius r.

Remark 3.17. Note that a metric space is weakly doubling exactly when for any two radii R and r we have a global upper bound to the number of elements of an r-net in any ball with radius R.

Theorem 3.18. *A weakly doubling metric space has a unique growth rate i.e. all of its nets have equivalent growth rates.*

Proof. Suppose first that the space X is a bounded weakly doubling metric space. Now taking any ε-net N of X and apply the definition of a weakly bounded metric space to the radii $d(X) + 1$ and ε we see that the net N has to be finite. Thus the growth function of any net in X is bounded from above and thus equivalent to the constant function $r \mapsto 1$ and the claim holds. This means that we can assume the metric space in question to be unbounded.

Let X be an unbounded weakly doubling metric space, and P and S its nets with net-constants ε_P and ε_S, respectively. By theorem 3.15 we now that there exists a ε_V-net V in X with $\mathrm{Ord}(V) \leq \mathrm{Ord}(S)$, $\mathrm{Ord}(V) \leq \mathrm{Ord}(S)$ and $\varepsilon_V \geq 4\max(\varepsilon_S, \varepsilon_P)$. We can even assume V to have a common point with both P and S by constructing V as the maximal ε_V separated subset of X containing certain fixed points $p \in P$ and $s \in S$ with $d(p, s) \geq \varepsilon_V$ (such points p and s can be found as the nets are full in the unbounded metric space X). To prove the claim it will be enough to show that $\mathrm{Ord}(V) \geq \mathrm{Ord}(S)$ and $\mathrm{Ord}(V) \geq \mathrm{Ord}(P)$. The claim is symmetric with respect to these nets so we will just show that $\mathrm{Ord}(V) \geq \mathrm{Ord}(S)$.

We take a base point $x_0 \in S \cap V$. As X is weakly doubling, there exists a constant K such that there can be at most K disjoint balls with radius ε_S in a ball with radius ε_V. This especially tells us that for any $x \in X$ we have that $\sharp\left(\mathrm{B}\left(x, \varepsilon_V\right) \cap S\right) \leq K$. Thus

$$\Gamma_S^{x_0}(r) \leq \sum_{x \in \mathrm{B}_V\left(x_0, r+\varepsilon_S\right)} \sharp\left(\mathrm{B}\left(x, \varepsilon_V\right) \cap S\right) \leq K \cdot \Gamma_V^{x_0}(r + \varepsilon_S),$$

and so $\Gamma_S^{x_0}(r) \leq K\Gamma_V^{x_0}(2r)$ whenever $r \geq \varepsilon_S$. □

Theorem 3.19. *Suppose a metric space X has a net S that is weakly doubling as a metric space. Then $\mathrm{Ord}(S)$ is the minimal element of $\mathcal{GR}(X)$.*

Proof. Assume there exists a weakly doubling net S in X. If we take any net P of X, we have by theorem 3.15 a net V in X with $\mathrm{Ord}(V) \leq \mathrm{Ord}(S)$ and $\mathrm{Ord}(V) \leq \mathrm{Ord}(P)$. But by theorem 3.14 we can pick from S a subnet S' such that $\mathrm{Ord}(S') \leq \mathrm{Ord}(V)$. But as S' is a net of S, we have by theorem 3.18 that $\mathrm{Ord}(S) = \mathrm{Ord}(S')$, and so $\mathrm{Ord}(S) = \mathrm{Ord}(V) \leq \mathrm{Ord}(P)$. □

Corollary 3.20. *Suppose a metric space X contains a full weakly doubling subset. Then there exists a minimal element in $\mathrm{Ord}_*(X)$.*

From now on if X is a metric space that has a minimal element in its growth class we will denote $\mathrm{Ord}(X) := \min \mathrm{Ord}_*(X)$.

Corollary 3.21. *Every bounded metric space has the growth rate $\mathcal{O}(x \mapsto 0)$.*

Proof. This follows immediately as we pick a growth net containing only a single point. □

We now look at how to get estimates for, or even calculate, the growth rate of a product of metric spaces.

Theorem 3.22. *Let X and Y be two metric spaces. For any nets $S \subset X$, $P \subset Y$ the set $S \times P$ contains a net V in $X \times Y$ such that $\mathrm{Ord}(V) \leq \mathcal{O}(\Gamma_S \cdot \Gamma_P)$.*

Proof. For our uses the most natural product metric will be

$$d_{X\times Y}((x, y), (x', y')) := \max(d_X(x, x'), d_Y(y, y')).$$

It is bi-Lipschitz equivalent to the more standard metric given by sums, so claims are equivalent for these different metrics. With this metric we note that if the net constants of S and P are ε_S and ε_P, respectively, then $S \times P$ is $\min(\varepsilon_S, \varepsilon_P)$-separated and $\max(2\varepsilon_S, 2\varepsilon_P)$-full in $X \times Y$. As it is full, it especially contains a net V that is a net also in $X \times Y$.

Fix a point $(x_0, y_0) \in V$. We note that the amount of points of V in any ball must be less than the amount of points of $S \times P$ in the same ball. On the other hand

$$(\mathrm{B}\left((x_0, y_0), r\right) \cap S \times P) = (\mathrm{B}\left(x_0, r\right) \cap S) \cdot (\mathrm{B}\left(y_0, \cap P\right)).$$

Combining these we see that

$$\Gamma_V^{(x_0,y_0)}(r) \leq \Gamma_{S\times P}^{(x_0,y_0)}(r) = \Gamma_S^{x_0} \cdot \Gamma_P^{y_0}(r)$$

and this proves the claim. □

Corollary 3.23. *Let X and Y be metric spaces.*

(1.) If there exists weakly doubling nets in X and Y, the product of these is weakly doubling full subset of $X \times Y$, and $\mathrm{Ord}(X \times Y) = \mathrm{Ord}(X) \times \mathrm{Ord}(Y)$.

(2.) If X and Y are weakly doubling, then so is $X \times Y$, and $\mathrm{Ord}(X \times Y) = \mathrm{Ord}(X) \times \mathrm{Ord}(Y)$.

Proof. To prove these claims it will suffice show that the product of two weakly doubling metric spaces is weakly doubling. But this follows easily when we use the same metric as in the proof of the previous theorem as in this metric balls of the product space are always products of balls of the factors. So the claim holds true. □

Corollary 3.24. *Let X and Y be metric spaces. If Y is bounded, then*

$$\mathrm{Ord}_*(X \times Y) = \mathrm{Ord}_X .$$

Theorem 3.25. *The euclidean space $\mathbb{R}^n$ is weakly doubling and* $\mathrm{Ord}(\mathbb{R}^n) = \mathcal{O}(x^n)$.

Proof. Note that all balls in Euclidean space are measurable in the sense of Lebesgue. Furthermore every ball has a finite measure, and as the Lebesgue measure is translation invariant the measure depends only on the radius. As the space $\mathbb{R}^n$ is separable, any collection of disjoint open sets must be countable. Especially if we have a collection $\mathcal{B}$ of disjoint balls with radius r within a ball of radius R in $\mathbb{R}^n$ it has to be countable collection. Denote by C_r the measure of a ball with radius $r > 0$. Now

$$C_R = m(\mathrm{B}\left(x_0, R\right)) \geq m\left(\bigcup_{B \in \mathcal{B}} B\right) = \sum_{B \in \mathcal{B}} m(B) = \sum_{B \in \mathcal{B}} C_r = C_r \cdot \sharp\mathcal{B},$$

so we must have

$$\sharp\mathcal{B} \leq \frac{C_R}{C_r} := K < \infty.$$

This proves the claim. □

3.2.3 Metric structure and growth rate of groups

Compact manifolds with path-length structure will be the most concrete examples of the objects for which we will formulate our results. Such manifolds, along with manifolds that in some sense look enough like compact manifolds, will always have a finitely generated fundamental group as we shall later see. Thus the concept of a finitely generated group arises quite naturally in this thesis. Moreover, the manipulation of growth concepts of a general metric space gets a bit technical. What will be extremely pleasant is that finitely generated groups are actually weakly doubling, so they have a unique growth rate. This is useful as the fundamental groups of our manifolds will hold considerable amount of the information we use in our results.

Let G be a group. We say that a set $S \subset G$ *generates the group G* if every element of G can be expressed as a finite combination of the elements of the set $S \cup S^{-1}$, where $S^{-1} := \{s^{-1} \in G \mid s \in S\}$. There always exists a basis for a group, namely the group itself. We are interested mostly of the cases in which the generating set, which we also call the spanning set, can be chosen to be finite. We say that a group G is *finitely generated*, if there exists a finite set that generates the group. A non-finitely generated group is a group that is not generated by any of its finite subsets.

Definition 3.26. Let G be a group generated by a set $S := \{g_i\}$. We define a norm

$$\|\cdot\|_S : G \to \mathbb{N} \subset \mathbb{R}_+$$

on this group by setting $\|e\|_S = 0$, and for $g \neq e$ we set $\|g\|_S$ to equal the infimum of the natural numbers k such that g can be written as combination of k elements of the set $S \cup S^{-1}$.

We also define a metric d_S, called *the word length metric* on this group by setting

$$d_S(g,h) = \left\|g^{-1}h\right\|_S$$

for all $g,h \in G$. We note that for the identity element e of G we have that $d_S(e,g) = \|g\|_S$ for all $g \in G$.

We note that with respect to any spanning set S the metric space (G, d_S) is 1-separated. This means that we can talk about the growth function Γ_G^S of a group. The selection of a base point in the definition of a growth function is not essential as we have already noted, but usually we will use the identity element as a base point.

Remark 3.27. Because a non-empty subset of natural numbers always contains a smallest element, the norm given in the previous definition is always obtained with some finite product of elements of S.

More precisely, if $S = \{g_i\}$ is a set of generators, we will often write an element g of the group with $\|g\|_S = k$ in the form

$$g_{i_1}^{\varepsilon^g(i_1)} \cdot \ldots \cdot g_{i_k}^{\varepsilon^g(i_k)},$$

when we want to give a specific representation of the element. In this representation the element-dependent symbols $\varepsilon^g(i_j)$ are ± 1. We leave ourselves the permission to omit the superscripts of these symbols when they are not needed for clarity in order to simplify notation.

Theorem 3.28. *The word length metric is a metric.*

Proof. We need to show three conditions on this function. All its values are clearly non-negative as natural numbers. Let g, h and s be arbitrary elements of G.

We first note that $d_S(g,g) = \left\|g^{-1}g\right\|_S = \|e\|_S = 0$.

Now we look at symmetry. If $d(g,h) = k$, then

$$g^{-1}h = g_{i_1}^{\varepsilon(i_1)} \cdot \ldots \cdot g_{i_k}^{\varepsilon(i_k)}.$$

Thus

$$h^{-1}g = \left(g^{-1}h\right)^{-1} = \left(g_{i_1}^{\varepsilon(i_1)} \cdot \ldots \cdot g_{i_k}^{\varepsilon(i_k)}\right)^{-1} = g_{i_k}^{-\varepsilon(i_k)} \cdot \ldots \cdot g_{i_1}^{-\varepsilon(i_1)},$$

and so $d_S(h,g) \le k = d_S(g,h)$. By symmetry of the argument, $d_S(h,g) = d_S(g,h)$.

The triangle inequality also follows quite mechanically, as if $d_S(g,h) = k$ and $d(h,s) = n$, then

$$g^{-1}h = g_{i_1}^{\varepsilon^1(i_1)} \cdot \ldots \cdot g_{i_k}^{\varepsilon^1(i_k)} \text{ and } h^{-1}s = g_{j_1}^{\varepsilon^2(j_1)} \cdot \ldots \cdot g_{j_n}^{\varepsilon^2(j_n)}.$$

Thus

$$g^{-1}s = \left(g^{-1}hh^{-1}\right)s = \underbrace{g_{i_1}^{\varepsilon^1(i_1)} \cdot \ldots \cdot g_{i_k}^{\varepsilon^1(i_k)} g_{j_1}^{\varepsilon^2(j_1)} \cdot \ldots \cdot g_{j_n}^{\varepsilon^2(j_n)}}_{k+n \text{ pcs.}},$$

and so

$$d_S(g,s) \le k+n = d_S(g,h) + d_S(h,s).$$

Thus the word length metric is a metric. □

For groups we see that the definitions of the norm, the word length metric and thus the growth function and -rate of a finitely generated group depend on the selected finite set of generators. We shall now see that they will nevertheless be equivalent for different sets of generators in the sense of asymptotic behaviour i.e. growth rate.

Theorem 3.29. *Let S_1, S_2 be two finite sets of generators for a finitely generated group G. Then there exists real constants a and b such that*

$$a \cdot \|g\|_{S_2} \le \|g\|_{S_1} \le b \cdot \|g\|_{S_2}$$

holds for all $g \in G$. So the word-norms in a group with respect to different spanning sets are always bi-Lipschitz equivalent. Especially the growth rate of the group does not depend on the finite set of generators.

Proof. We shall prove only the second inequality for the first inequality follows from it.

As the set S_1 generates the group G, we may write each element g_i of S_2 as a combination of k_i elements of S_1. As the set S_2 was finite, we may pick the largest of the numbers k_i, call it k. Now if we pick any $g \in G$ with $\|g\|_{S_2} = n$ and write it as a combination of n elements of S_2, we can replace each g_i in this representation by a product of at most k elements of the set S_1. Thus we get a representation for g with at most $kn = k \cdot \|g\|_{S_2}$ elements of S_1, so

$$\|g\|_{S_1} \le k \cdot \|g\|_{S_2}.$$

Thus the claim holds true. □

We especially see that for finitely generated groups the rapidity of the growth of a growth function Γ_G^S does not depend on the selected finite set of generators. This means that we can write Γ_G when we are only interested in the asymptotic behaviour of Γ^S.

As we noted earlier, the growth concepts of a metric space turn much simpler if our space is weakly doubling. We now show that finitely generated groups are always weakly doubling. Furthermore we note that for any generating set S for a group G we have that

$$\mathrm{B}\left(e, n\right) \supset \mathrm{B}\left(e, 1\right) = S \cup S^{-1} \cup \{e\}. \quad \text{Thus} \quad \sharp\, \mathrm{B}\left(e, n\right) \geq \sharp\, \mathrm{B}\left(e, 1\right) \geq \sharp S$$

for all $n \geq 1$. So if we have a non-finitely generated group, it has to have infinitely many elements in any ball centered at the neutral element. A group is always 1-separated, so this tells us that a non-finitely generated group cannot be weakly doubling. This means that we could easily reformulate the following result in a stronger form that would say that a group (G, d_S) is weakly doubling exactly when S is finite.

Lemma 3.30. *Finitely generated groups are weakly doubling.*

Proof. Note that $\mathrm{B}\left(g, R\right) = g \cdot \mathrm{B}\left(e, R\right)$, so $\sharp(\mathrm{B}\left(g, R\right)) = \sharp(\mathrm{B}\left(e, R\right)))$. Any ball within $\mathrm{B}\left(e, R\right)$ with radius $r \geq 0$ must contain at least one point, so in any ball in G with radius R there can be at most $\sharp\, \mathrm{B}\left(e, R\right)$ disjoint balls with radius $r < R$. Assume now that the group is finitely generated, with a generating set S, $\sharp S = K$. Now for any $n \in \mathbb{N}$

$$\sharp\, \mathrm{B}\left(e, n+1\right) \leq K \sharp\, \mathrm{B}\left(e, n\right) \leq \cdots \leq K^n \sharp\, \mathrm{B}\left(e, 1\right) = K^n.$$

This proves the claim. $\square$

Thus the set $\mathrm{Ord}_*(G)$ always has a minimal element $\mathrm{Ord}(G)$ when G is finitely generated.

Example 3.31. Let us look at the growth functions and orders of growths of some familiar groups and metric spaces.

1. If we choose the set $\{1\}$ as a finite generator for the group $(\mathbb{Z}, +)$, we see that

   $$\Gamma_{\mathbb{Z}}(n) = \sharp\{\underbrace{\pm 1 \pm \ldots \pm 1}_{k \text{ pcs.}} \mid 0 \leq k \leq n\} = \sharp\, \{-n, \ldots, n\} = 2n + 1.$$

   From this we see that the order of growth of this group is linear, $\mathrm{Ord}(\mathbb{Z}) = \mathcal{O}(x^1)$.

   If we would choose another set of generators, say $\{1, 2\}$, we would see that with respect to this set of generators of $(\mathbb{Z}, +)$ we have that

   $$\Gamma'_{\mathbb{Z}}(n) = \sharp\, \{-2n, \ldots, 2n\} = 4n + 1,$$

   and again $\mathrm{Ord}(\mathbb{Z}) = \mathcal{O}(x^1)$. So even though a growth function changes, the order of growth of the group stays the same due to the bi-Lipschitz invariance.

2. All finite groups have polynomial growth with order of growth at most 0. This can be seen by noting that for a finite group G we can take the group itself as a generator. Now $\Gamma_G(n) = \sharp G$ for all $n \geq 2$, and so $\mathrm{Ord}(G) = \mathcal{O}(x^0)$.

3. Let H be a finitely generated[3] subgroup of a finitely generated group G. By expanding any finite set of generators of H to a set of generators of G we easily see that $\operatorname{Ord}(G) \geq \operatorname{Ord}(H)$.

4. Free group of 2 generators has exponential growth. Suppose the free group is spanned by two symbols a and b. If we take for this the finite set $\{a, b\}$ as a set of generators we see that

$$\Gamma(n) = 2^{2n} + 1,$$

so $\operatorname{Ord}(\langle a, b\rangle) = \mathcal{O}(n \mapsto e^n)$.

5. The set $\mathbb{Z}^2$ with metric induced from $\mathbb{R}^2$ is bi-Lipschitz equivalent to the abelian group $(\mathbb{Z}^2, +)$ with the finite set of generators $\{(0,1), (1,0)\}$. It also has growth rate of order 2. This can be seen from the fact that any ball $\mathrm{B}\big((0,0), r\big) \subset \mathbb{Z}^2$ is contained in a cube with side length $2r$ and contains a cube with side length $\sqrt{2}r$. (This holds with respect to either of the metrics.) Thus

$$2n^2 \leq \Gamma_{\mathbb{Z}^2}^{(0,0)}(n) \leq 4n^2$$

for all $n \in \mathbb{N}$.

In general it can be hard to find groups with a given growth rate, and it is still somewhat unknown what kind of growth rates can groups have. We mention a celebrated result of Grigorchuk that states that there exists a group with superpolynomial but sub-exponential growth [Gri84].

3.3 Cayley graphs of groups

We next look at a way to visualize (finitely generated) groups and their metrics with respect to different spanning sets. We do this by introducing the concept of a *Cayley graph*. Each group with a set of spanning elements can be identified with a graph that has the group elements as edges and translated spanning elements as vertexes.

Definition 3.32. Let G be a group and $S \subset G$ a set that generates this group. The *Cayley graph* $C(G, S)$ of this pair is defined by selecting the set G as the edges and defining the edges as the set of pairs $(g, h) \in G \times G$ such that $g^{-1}h \in \big(S \cup S^{-1}\big) \setminus \{e\}$.

This means that there exists a vertex between points g and h if and only if one can be acquired from the other by multiplying with a spanning element or its inverse. We omit here the neutral element from the spanning set in order to remove edges from a vertex to itself.

Some examples of Cayley graphs of familiar groups are shown in figures 5, 6 and 7.

We define *paths* on a graph to be a finite ordered collection

$$\{(a_i, b_i) \subset G \times G \mid b_i = a_{i+1}, i = 1, \ldots, n\}$$

[3]Perhaps surprisingly, subgroups of finitely generated groups need not be finitely generated. For example the commutator of the free group of two elements is not finitely generated. This follows from example from [Coh89, Corollary 3, p. 15].

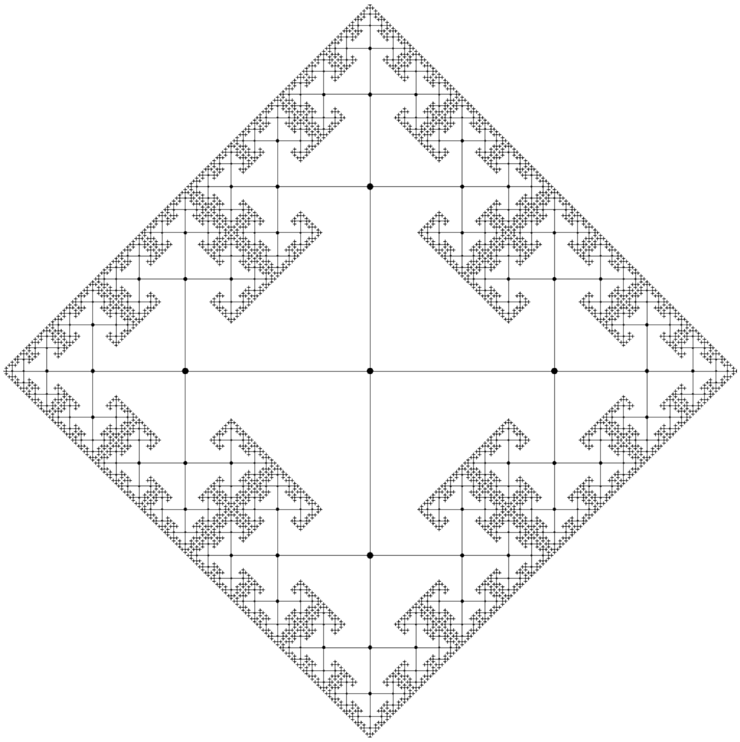

Figure 5: The Cayley graph of $(\langle a, b\rangle, \{a, b\})$.

of vertexes such that two consecutive vertexes have a common point as their respective endpoint and beginning point.

The length $\ell(\gamma)$ of a path γ is just the number of elements in it as a set. With this we can define a length-metric[4] on a graph.

Definition 3.33. The length-metric on a graph is

$$d(x, y) = \inf \{\ell(\gamma) \mid \gamma \text{ is a path and the endpoints of } \gamma \text{ are } x \text{ and } y\}.$$

The group metric of a finitely generated group and the metric of the corresponding Cayley graph are essentially same in the following sense.

Theorem 3.34. *The natural bijection*

$$i\colon (G, d_S) \to (C(G, S), d), \quad g \mapsto g$$

is an isometry.

Proof. The theorem is clear as the definitions of the two metrics are practically identical. ☐

[4]This actually corresponds to certain degree with a so called *length structure* as will be defined in definition 4.1.

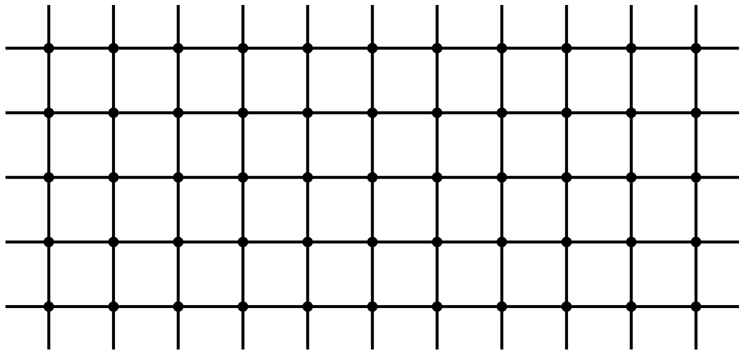

Figure 6: Part of the Cayley graph of $\left(\mathbb{Z}^2, \{(1,0),(0,1)\}\right)$.

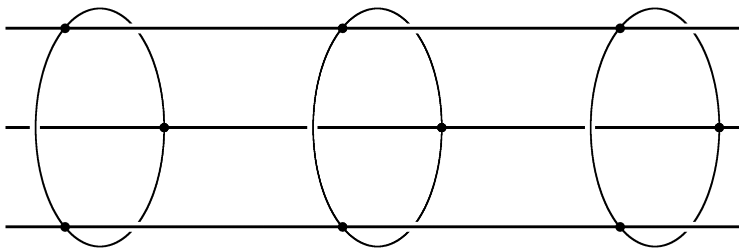

Figure 7: Part of the Cayley graph of $(\mathbb{Z} \times \mathbb{Z}_3, \{(1,[0]_3),(0,[1]_3)\})$.

3.4 Lipschitz quotient- and coarse Lipschitz quotient mappings

We now turn our attention into combining our knowledge of coarse quasi-isometries and growth rate. To accomplish this it is natural at this point to introduce the concept of a (coarse) Lipschitz quotient map. We give the basic definition and properties required in this thesis, for further information we refer to [BJL+99].

We show first that coarse quasi-isometries are always coarse Lipschitz quotients, and then prove that Lipschitz quotient mappings cannot increase the growth rate, at least in some sense. After that we will easily see that coarse quasi-isometries have to preserve growth rates in the weak way that coarse Lipschitz quotient maps do not increase them. Later in section 5 we will show that BLD mappings are Lipschitz quotients when the domain is complete, and with this we will be able to get restraints to their existence.

Definition 3.35. A mapping $f\colon X \to Y$ between two metric spaces is called *Lipschitz quotient* if there exists constants $0 < C_1 \leq C_2$ and $r_0 \geq 0$ such that for any $x \in X$ we have that

$$\mathrm{B}\left(f(x), C_1 r\right) \subset f\left[\mathrm{B}\left(x, r\right)\right] \subset \mathrm{B}\left(f(x), C_2 r\right)$$

for all $r \geq r_0$. Note that a Lipschitz quotient map has to be surjective by the first inclusion.

Example 3.36. Let us look at some examples.

1. All bi-Lipschitz maps are Lipschitz quotient.
2. The floor mapping $\mathbb{R} \to \mathbb{Z}$, $x \mapsto \lfloor x \rfloor$ is Lipschitz quotient, but not continuous.
3. The mapping $\mathbb{R}_+ \to \mathbb{R}_+$, $x \mapsto x^2$ is a homeomorphism, but not a Lipschitz quotient map.
4. We shall show later (theorem 5.14) that all BLD mappings between path-length manifolds are Lipschitz quotients when the domain is complete.
5. The inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$ is a coarse quasi-isometry but not a Lipschitz quotient map.
6. The mapping $f\colon \mathbb{Z} \to \mathbb{N}$ defined by $f(k) = |k|$ is a Lipschitz quotient map.

Lemma 3.37. *The composition of two Lipschitz quotient mappings is a Lipschitz quotient.*

Proof. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two Lipschitz quotient maps and $x_0 \in X$ a point. For any $r > \max(r_f, r_g)$ we see that

$$\begin{aligned}(g \circ f)\left[\mathrm{B}\left(x_0, r\right)\right] &= g\left[f\left[\mathrm{B}\left(x_0, r\right)\right]\right] \\ &\subset g\left[\mathrm{B}\left(f(x_0), C_f r\right)\right] \\ &\subset \mathrm{B}\left(g(f(x_0)), C_g C_f r\right) \\ &= \mathrm{B}\left((g \circ f)(x_0), C_g C_f r\right).\end{aligned}$$

The other inclusion is similar. $\square$

We would like to compose Lipschitz quotients with coarse quasi-isometries, but such compositions do not usually result in a Lipschitz quotient. This is why we define a more flexible concept called a coarse Lipschitz quotient mapping.

Definition 3.38. A mapping $f\colon X \to Y$ between two metric spaces is called *coarse Lipschitz quotient* if $f\colon X \to f[X]$ is Lipschitz quotient and the set $f[X]$ is ε-full in Y.

Example 3.39. There is no Lipschitz quotient map from natural numbers to the integers.

Theorem 3.40. *Coarse quasi-isometries are coarse Lipschitz quotients.*

Proof. Let $f\colon X \to Y$ be a coarse quasi-isometry. The image of a coarse quasi-isometry f is ε-full in Y by definition, so we only need to check the inclusions concerning balls.

As f is assumed a coarse quasi-isometry, we have constants $C > 0$, $D \geq 0$ such that

$$C^{-1}d(x,y) - D \leq d(f(x), f(y)) \leq Cd(x,y) + D$$

for all points $x, y \in X$. Let $x_0 \in X$ and $y \in f[\mathrm{B}\left(x_0, r\right)]$ with $x \in \mathrm{B}\left(x_0, r\right)$ such that $f(x) = y$. We note that

$$d(y, f(x_0)) = d(f(x), f(x_0)) \leq Cd(x, x_0) + D < Cr + D,$$

so $y \in \mathrm{B}\left(f(x_0), Cr + D\right)$. This means that for all $r \geq D$ we have that

$$f[\mathrm{B}\left(x_0, r\right)] \subset \mathrm{B}\left(f(x_0), (C+1)r\right).$$

Let now $z \in \mathrm{B}\left(f(x_0), R\right) \cap f[X]$. This means that there exists x in X such that $f(x) = z$. Now

$$C^{-1}d(x, x_0) - D \leq d(f(x), f(x_0)) = d(z, f(x_0)) < R,$$

so $x \in \mathrm{B}\left(x_0, C(R+D)\right)$. This holds for all $R > 0$, so by denoting $r = C(R+D)$ we see that

$$z \in \mathrm{B}\left(f(x_0), C^{-1}r - D\right) \cap f[X]$$

implies $f^{-1}\{z\} \subset \mathrm{B}\left(x_0, r\right)$ for all $r > D/C$. But this implies that

$$\mathrm{B}\left(f(x_0), (2C)^{-1}r\right) \subset f[\mathrm{B}\left(x_0, r\right)]$$

for all $r > \max(D/C, 2D)$ and this completes the proof. □

Remark 3.41. By combining previous theorem and the latest example we now see that there cannot exist a coarse quasi-isometry between $\mathbb{Z}$ and $\mathbb{N}$ even though there exist a coarse Lipschitz quotient from $\mathbb{Z}$ to $\mathbb{N}$ as was also seen in the latest example. Thus these classes of mappings are strictly different.

The following is a corollary of lemma 3.4 and theorem 3.40.

Corollary 3.42. *Let X and Y be metric spaces, $f\colon X \to Y$ a coarse Lipschitz quotient mapping between them and*

$$\xi_X\colon X' \to X \quad \text{ and } \quad \xi_Y\colon Y' \to Y$$

coarse quasi-isometries. Then $\xi_Y^{-1} \circ f \circ \xi_X$ is a coarse Lipschitz quotient.

Lemma 3.43. *Let X be a metric space. Suppose A is a full subset of X and B is a full subset of A. Then B is full in X.*

Proof. The claim follows immediately by triangle inequality. □

To prove some of the following important results we will need the following lemma.

Lemma 3.44. *Let $f\colon X \to Y$ be a coarse Lipschitz quotient map and $A \subset X$ a full subset of X. Then $f[A]$ is full in Y.*

Proof. By the previous lemma it will suffice to prove the claim just for Lipschitz quotient mappings.

Let $y \in Y$. As Lipschitz quotient mappings are always surjective there exists a point $x \in X$ such that $f(x) = y$. Because $A \subset X$ is full in X we have a constant ε such that $\mathrm{B}\left(A, \varepsilon\right) = X$. This especially means that there exists $a \in A$ with $d(x, a) < \varepsilon$.

As f was assumed Lipschitz quotient, we have constants C and r_0 such that

$$\mathrm{B}\left(f(x), \frac{r}{L}\right) \subset f[\mathrm{B}\left(x, r\right)] \subset \mathrm{B}\left(f(x), Lr\right)$$

for all points $x \in X$ whenever $r \geq r_0$. Denote $R := \max\{\varepsilon, r_0\}$. As $x \in \mathrm{B}\left(a, R\right)$, we must have

$$f(x) \in f[\mathrm{B}\left(x, R\right)] \subset \mathrm{B}\left(f(x), LR\right),$$

so $f[A]$ is LR-full in Y. □

Lemma 3.45. *The composition of two coarse Lipschitz quotient mappings is coarse Lipschitz quotient.*

Proof. The inclusions are proven exactly as in lemma 3.4. Suppose f and g are mappings whose images are full. Now $\mathrm{Im}(f \circ g) = f[\mathrm{Im}(g)]$. As g is coarsely Lipschitz quotient its image is full, and by previous lemma the image of a full set under a coarse Lipschitz quotient map is full. □

Example 3.46. 1. All Lipschitz quotient maps are of course coarse Lipschitz quotient.

2. The inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$ is not Lipschitz quotient because it is not surjective. It is coarse Lipschitz quotient as a coarse quasi-isometry.

3. The mapping $\mathbb{Z} \to \mathbb{R}_+$, $k \mapsto |k|$ is coarse Lipschitz quotient but not a coarse quasi-isometry.

All of our results that connect different classes of mappings to the growth rate will be based on the following theorem.

Theorem 3.47. *Suppose we have two metric spaces X and Y with $f\colon X \to Y$ a coarse Lipschitz quotient map. For any net S of X there exists a net $S' \subset f[S]$ of Y such that* $\mathrm{Ord}(S) \geq \mathrm{Ord}(S')$.

Proof. Let S be an ε_S-net of X, and let $f\colon X \to Y$ be a coarse Lipschitz quotient mapping with constants C_1, C_2, r_0 and ε_f such that

$$\mathrm{B}\left(f(x), C_1 r\right) \subset f\left[\mathrm{B}\left(x, r\right)\right] \subset \mathrm{B}\left(f(x), C_2 r\right)$$

holds for all $x \in X$, $r \geq r_0$ and $\mathrm{B}\left(\mathrm{Im}(f), \varepsilon_f\right) = Y$.

The net S is full in X, so by lemma 3.44 its image $f[S]$ under the coarsely Lipschitz quotient map f is full in Y. This means that by lemma 3.12 there exists a net S' in $f[S]$ that is a net also in Y. To prove our claim we only need to show now that $\mathrm{Ord}(S) \geq \mathrm{Ord}(S')$. Let us fix a point $y_0 \in S'$ and pick $x_0 \in S \cap f^{-1}\{y_0\}$. For any ball $\mathrm{B}\left(x_0, r\right)$ we note that by Lipschitz quotient inequalities we have

$$f[\mathrm{B}\left(x_0, r\right)] \supset \mathrm{B}\left(y_0, \frac{r}{L}\right)$$

for all $r \geq r_0$. By definition of the net S', we note that

$$f[S \cap \mathrm{B}\left(x_0, r\right)] \supset S' \cap \mathrm{B}\left(y_0, \frac{r}{L}\right).$$

For any set A we always have $\sharp A \geq \sharp f[A]$, so we get finally that

$$\sharp\left(S \cap \mathrm{B}\left(x_0, r\right)\right) \geq \sharp f[S \cap \mathrm{B}\left(x_0, r\right)] \geq \sharp\left(S' \cap \mathrm{B}\left(y_0, \frac{r}{L}\right)\right).$$

This means that $\Gamma_S(r) \geq \Gamma_{S'}(r/L)$ for all $r \geq r_0$, which proves the claim.

□

Remark 3.48. Note that theorem implies that if we have coarse Lipschitz quotient mapping between metric spaces X and Y, then any lower bound of the growth class of Y gives a lower bound for the growth class of X.

Now we are ready to prove the following theorem which is, combined with the remark that follows it, the most important property of coarse quasi-isometries. Growth rate is a coarse quasi-isometry invariant in the following sense.

Theorem 3.49. *Suppose we have two metric spaces X and Y with $f\colon X \to Y$ a coarse quasi-isometry. For any net S of X there exists a net S' of Y such that $\operatorname{Ord}(S) \geq \operatorname{Ord}(S')$, and for any net P of Y there exists a net P' of X such that $\operatorname{Ord}(P) \geq \operatorname{Ord}(P')$.*

Proof. The first claim follows by theorem 3.47, to the function f which is by theorem 3.40 coarse Lipschitz quotient. The second claim follows by doing the same to the coarse quasi-isometry f^{-1}. □

Remark 3.50. Note that theorem implies that if we have coarsely quasi-isometric metric spaces X and Y, then any lower bound of the growth class of Y gives a lower bound for the growth class of X as well and vice versa.

Moreover, if both growth classes contain a minimal element, they must equal. This is what is we mean by coarse quasi-isometries preserving growth of metric spaces.

Remark 3.51. Note that theorem 3.49 is not of the form 'if and only if'. For example the spaces $\mathbb{Z}$ and $\mathbb{N}$ both have polynomial order of growth 1, but they are not coarsely quasi-isometric. (This was seen in one of the previous examples.)

4 Length space manifolds

To state and prove the main result of this thesis we need most of all a path-metric (and a length structure) to our manifold. We prove all of our main results for general topological manifolds with length structure, but to give concrete examples we show that Riemannian- and Lipschitz manifolds are such objects.

We will also need to bound the geometric behaviour of our manifolds. We take one form of the usual requirement in this field and give upper and lower bounds to the 'sizes of holes' in our manifold. In some of our more sophisticated results we use stronger assumptions, most of which are kind of homogeneity requirements on the geometry. We will give some stronger, more concrete requirements that will guarantee manifolds to have the needed criteria. Most importantly we will see that compact manifolds have all the properties that bind fundamental group in the ways that we define in this section. We note that when proving the existence of a universal cover for our manifold, we showed that any manifold is (semi)locally simply connected. This requirement can be seen to be a sort of lower bound on the size of holes in our manifold, but is too purely topological in nature to give us a bound rigid enough.

4.1 Path-metric structures

We begin by defining a length structure on a set. This definition is from [Gro99, definition 1.3., p.2].

Definition 4.1. A *length structure* $(\mathcal{C}, \ell)$ on a set X is a collection of paths $\mathcal{C}(I)$ in X for each interval $I \subset \mathbb{R}$ and a mapping $\ell\colon \mathcal{C} := \bigcup \mathcal{C}(I) \to \mathbb{R}_+$ having the following properties:

(a) All constant paths belong to $\mathcal{C}$, and for $f \in \mathcal{C}$ we have that $\ell(f) = 0$ if and only if f is a constant path.

(b) If $I \subset J$, and $f \in \mathcal{C}(J)$, then $f|_I \in \mathcal{C}(I)$.

(c) If $f \in \mathcal{C}([a, b])$ and $g \in \mathcal{C}([b, c])$ with $f(b) = g(b)$, the path $f * g\colon [a, c] \to X$ constructed naturally from f and g belongs to $\mathcal{C}([a, c])$ and we have that $\ell(f * g) = \ell(f) + \ell(g)$.

(d) If $\varphi\colon I \to J$ is a homeomorphism and $f \in \mathcal{C}(J)$, then $f \circ \varphi \in \mathcal{C}(I)$, and $\ell(f \circ \varphi) = \ell(f)$.

(e) For each $f \in \mathcal{C}([a, b])$, the map $t \mapsto \ell(f|_{[a,t]})$ is continuous.

We call the elements of $\mathcal{C}$ *rectifiable paths*, and we set $\ell(\gamma) = \infty$ for any non-rectifiable paths. If we have a length-structure $\mathcal{C}$ on a set X such that any two points in X can be connected with a rectifiable path we say that the set X is rectifiably connected with respect to $\mathcal{C}$. A *length manifold* $\mathcal{M}$ is a manifold together with a length structure $(\mathcal{C}, \ell)$ such that $\mathcal{M}$ is rectifiably connected with respect to $\mathcal{C}$ and so that the topology given by the metric defined by

$$(1) \qquad d_\ell(x, y) = \inf \{\ell(\gamma) \mid \gamma \in \mathcal{C}, \gamma\colon x \curvearrowright y\}$$

coincides with the original topology of our manifold. We must have the manifold rectifiably connected to get a metric from the function d_ℓ. When talking about length manifolds we shall always refer by 'metric' to the path-metric given by (1).

We can equip any metric space with a path-length structure as follows.

Definition 4.2. Let (X, d) be a path-connected metric space. We define the length $\ell_d(\cdot)$ of a path $\gamma\colon [0,1] \to X$ by setting

$$\ell_d(\gamma) = \sup \left\{ \sum_{i=1}^{n-1} d\left(\gamma(a_i), \gamma(a_{i+1})\right) \mid a_1, \ldots, a_n \in [0,1], a_1, < \cdots < a_n, n \in \mathbb{N} \right\}$$

We will allow the values of ℓ_d be in $[0, \infty]$. We call paths with finite length rectifiable.

This path-length structure clearly satisfies the requirements in the definition of path-length structure. Do note that even when d_ℓ is a metric, it need not be (bi-Lipschitz) equivalent to the original metric d.

Theorem 4.3. *Suppose $\mathcal{M}$ is a manifold and $\widehat{\mathcal{M}}$ its cover. If $\mathcal{M}$ is a length manifold, so is $\widehat{\mathcal{M}}$. Furthermore the length structure can be chosen so that the covering map becomes a local isometry.*

Proof. Denote by $(\mathcal{C}, \ell)$ the length structure of $\mathcal{M}$. The length structure $(\widetilde{\mathcal{C}}, \widetilde{\ell})$ of universal cover is obtained by taking for each interval $I \subseteq \mathbb{R}$ all the continuous paths $\gamma\colon I \to \widehat{\mathcal{M}}$ for which $(p_{\mathcal{M}} \circ \gamma) \in \mathcal{C}(I)$, and defining $\widetilde{\ell}(\gamma) = \ell(p_{\mathcal{M}} \circ \gamma)$. This definition gives us a length structure in the sense of definition 4.1. We skim through the requirements.

Part (a) follows as the covering map is a local homeomorphism, so $p_{\mathcal{M}} \circ \gamma$ is a constant map if and only if γ is. So we see that

$$\widetilde{\ell}(\gamma) = 0 \quad \Leftrightarrow \quad \ell(p_{\mathcal{M}} \circ \gamma) = 0 \quad \Leftrightarrow \quad p_{\mathcal{M}} \circ \gamma \text{ is a constant path .}$$

Parts (b) and (c) are clear by the definition of $\widetilde{\mathcal{C}}(I)$.

To see part (d), let $\varphi\colon I \to J$ be a homeomorphism. Now because $\gamma \circ \varphi$ is rectifiable so is $(p_{\mathcal{M}} \circ \gamma) \circ \varphi$. Also

$$\widetilde{\ell}(\gamma \circ \varphi) = \ell(p_{\mathcal{M}} \circ \gamma \circ \varphi) = \ell(p_{\mathcal{M}} \circ \gamma) = \widetilde{\ell}(\gamma).$$

Part (e) follows as the covering map is continuous.

Furthermore the cover is rectifiably connected when $\mathcal{M}$ is. To see this we take any two points $x, y \in \widehat{\mathcal{M}}$. As the pre-images of covering neighbourhoods under $p_{\mathcal{M}}$ form a basis for the topology of $\widehat{\mathcal{M}}$, we may connect x and y with a finite sequence $U_1, \ldots, U_n$ of domains such that $p_{\mathcal{M}}[U_j]$ is a domain in $\mathcal{M}$ for all $j = 1, \ldots, n$. Within these domains we can construct rectifiable paths and connect the points x and y with a composition of these finitely many paths.

We can now define the path length metric in $\widehat{\mathcal{M}}$ as before with a length structure; we set

$$\widetilde{d}(x, y) = \inf \left\{ \widetilde{\ell}(\gamma) \mid \gamma\colon x \curvearrowright y, \gamma \text{ rectifiable } \right\}.$$

With the given definition of length in $\widehat{\mathcal{M}}$, the covering map $p_{\mathcal{M}}$ comes a local isometry. As it is both local isometry and local homeomorphism, the original topology of $\widehat{\mathcal{M}}$ coincides with the topology given by the metric $\widetilde{d}$. $\square$

From now on we assume all covers of length-manifolds to be also length-manifolds with the metric constructed in the previous theorem. Note that whenever we have a geodesic on a manifolds, any of its lifts is a geodesic in the cover. Composing a geodesic with a covering map gives rise to a local geodesic.

We shall later need the following structure result for complete length space manifolds. This can be seen to be a Hopf-Rinow type theorem. We prove the result first in full generality.

Theorem 4.4. *Complete locally compact connected length spaces have the Heine-Borel property, i.e. every closed bounded subset is compact.*

Proof. Let X be a complete locally compact connected length space. It suffices to show that closed balls are compact, as all bounded sets are by definition subsets of balls, and closed subsets of compact sets are compact. Let us assume that there exists a closed ball $\overline{\mathrm{B}}(x_0, R)$ that is not compact. By taking a small enough neighbourhood of x_0 we find by local compactness radius $r < R$ such that $\overline{\mathrm{B}}(x_0, r)$ is compact. This means that the set

$$\{r \in \mathbb{R}_+ \mid \overline{\mathrm{B}}(x_0, r) \text{ is compact}\} \subset \mathbb{R}$$

is non-empty and bounded above, so there exists a supremum of this set which we denote by R_0. We will first show that $\overline{\mathrm{B}}(x_0, R_0)$ is compact. Note that *all* closed balls centered at x_0 with radius strictly less than R_0 need to be compact, as the closed balls are closed sets and a closed subset of a compact set is itself compact.

Let us assume that $\overline{\mathrm{B}}(x_0, R_0)$ is not compact. Then there exists a sequence $(x_n) \subset \overline{\mathrm{B}}(x_0, R_0)$ that has no convergent subsequence. As the balls $\overline{\mathrm{B}}(x_0, r)$, $r < R_0$ are all compact, we must have that $\sharp\left((x_n) \cap \overline{\mathrm{B}}(x_0, r)\right) < \infty$ for all $r < R_0$ because otherwise we would find a converging subsequence from one of these compact balls. This especially means that there must be a subsequence of (x_n) in $\overline{\mathrm{B}}(x_0, R_0) \setminus \overline{\mathrm{B}}(x_0, r)$ for all $r < R_0$.

The sequence (x_n) is a sequence within a closed set of a complete metric space. If it had a Cauchy subsequence, this subsequence would converge in X and limit would lie in the closed set $\overline{\mathrm{B}}(X, x_0)$. But because we assumed that the sequence (x_n) has no converging subsequences this cannot happen, so the sequence (x_n) has no Cauchy subsequences. Especially there must be a constant $c > 0$ such that we have infinitely many elements of the sequence (x_n) with mutual distances at least c. Call this collection K.

Let us pick $r \in \left]R_0 - c/4, R_0\right[$. Denote

$$\mathcal{A} := \left\{\overline{\mathrm{B}}(x, c) \mid x \in \left(\overline{\mathrm{B}}(x_0, R_0) \setminus \overline{\mathrm{B}}(x_0, r)\right) \cap K\right\}.$$

Suppose $A := \overline{\mathrm{B}}(x, c) \in \mathcal{A}$. By connecting x_0 to x with a short enough path we find $x_A \in \partial\,\mathrm{B}\left(x_0, r\right)$ such that

$$A \cap \partial\,\mathrm{B}\left(x_0, r\right) \supset \mathrm{B}\left(x_A, \frac{c}{4}\right) \cap \partial\,\mathrm{B}\left(x_0, r\right).$$

Note that by previous deduction we must have $\sharp\mathcal{A} = \infty$ But this cannot happen as the set $\partial\,\mathrm{B}\left(x_0, r\right)$ is compact by assumption, so we find a finite subcover to an open cover consisting of balls with diameter with radius less than $c/8$, and from this finite set we have a surjection to the infinite set $\mathcal{A}$. This is a contradiction and thus the set $\overline{\mathrm{B}}(x_0, R_0)$ must be compact.

Let us now take open cover

$$\left\{\mathrm{B}\left(x, r\right) \mid x \in \partial\overline{\mathrm{B}}(x_0, R_0), \quad \overline{\mathrm{B}}(x, r) \text{ compact}, \quad r \in \mathbb{R}_+\right\}$$

for $\partial\overline{\mathrm{B}}(x_0, R_0)$. (This is possible as the metric space was assumed locally compact.) We pick for this a finite subcover $\mathcal{B}$. Note that the union

$$A := \overline{\mathrm{B}}(x_0, R) \cup \bigcup_{B \in \mathcal{B}} \overline{B}$$

is compact, as it is a union of finitely many compact sets. Now there exists a Lebesgue number $\lambda > 0$ for the cover $\mathcal{B}$ as it is an open cover of the compact set $\partial\overline{\mathrm{B}}(x_0, R)$. From this it follows that $\overline{\mathrm{B}}(x_0, R_0 + \lambda/2)$ is compact as a closed subset of the compact set A. This is a contradiction with the definition of R_0 and so we must have the claim. □

Corollary 4.5. *Complete length manifolds $\mathcal{M}$ with a path-metric have the Heine-Borel property, i.e. every closed bounded subset is compact.*

Corollary 4.6. *Suppose $\mathcal{M}$ is a complete length manifold and S is a net in $\mathcal{M}$. Then $\Gamma_S(r) < \infty$ for all $r \in \mathbb{R}_+$.*

Proof. Let S be on ε-net in the complete length manifold $\mathcal{M}$. For any $x_0 \in \mathcal{M}$ and $r \in \mathbb{R}_+$ we look at the collection

$$U := \{\mathrm{B}\left(x, \varepsilon/4\right) \mid x \in S \cap \overline{\mathrm{B}}(x_0, r).\}$$

This consists of disjoint balls as S was ε-separated. Let us take for $\overline{\mathrm{B}}(x_0, r + \varepsilon)$ an open cover consisting of balls with radius less than $\varepsilon/10$ and call it $\mathcal{B}$. For this we can by compactness guaranteed by corollary 4.5 take a finite subcover $\mathcal{A}$. Now every element in U contains an element of $\mathcal{A}$. As the elements of U were disjoint there now is a surjection from the finite set $\mathcal{A}$ to U and this proves the claim.

□

A length-space (X, d_ℓ) is called *geodesically complete* if for any two points $x, y \in X$ there exists a length-minimizing path $\gamma\colon x \underset{g}{\curvearrowright} y$. The following is also a Hopf-Rinow type theorem. Proof can be found for example from [Gro99, Section 1.12., p.9]. (The proof is essentially an application of Ascoli's theorem.)

Theorem 4.7. *A complete length-manifold is geodesically complete.*

4.2 Concrete examples of length manifolds

As promised, we now show that two familiar classes of manifolds are length manifolds. We give the definitions briefly and for further information and details we refer to [Lee03].

4.2.1 Riemannian manifolds

A *smooth structure* for a topological manifold $\mathcal{M}$ is a collection $\mathcal{A}$ of charts $(U_\alpha, \varphi_\alpha)$ of $\mathcal{M}$ such that the sets U_α cover the manifold $\mathcal{M}$ and the *transition mappings* $\left(\varphi_\alpha \circ \varphi_\beta^{-1}\right)|_{U_\alpha \cap U_\beta}$ are smooth mappings. A *smooth manifold* is a topological manifold with a maximal smooth structure.

Definition 4.8. A *Riemannian manifold* is a smooth manifold with a smooth 2-covariant tensor field g that is symmetric and positive definite. This tensor field is called the Riemann tensor. A Riemannian manifold $\mathcal{M}$ with the corresponding Riemann tensor is denoted by $(\mathcal{M}, g)$.

All smooth manifolds have a Riemann tensor, so we could only assume our manifolds to be smooth in this section, but this would be only an artificial enhancement.

Definition 4.9. Let $\gamma\colon [a,b] \to \mathcal{M}$ be a smooth path on a Riemannian manifold $(\mathcal{M}, g)$. We define the *length* of this path to be

$$\ell(\gamma) = \int_a^b g_{\gamma(t)}(\dot{\gamma}, \dot{\gamma})\ \mathrm{d}t.$$

The length of a piecewise path is defined to be the sum of the lengths of the smooth components.

In lemma 2.14 we found paths between arbitrary points of a manifold. In this proof the final path was a composition of finitely many paths that were "lifted" from $\mathbb{R}^n$ to our manifold by chart-homeomorphisms. In the smooth case we may imitate this proof by using *smooth* charts instead of mere homeomorphisms, and lift not just continuous paths but piecewise smooth paths. This is possible as each connected open subset of $\mathbb{R}^n$ is not only path-connected, but we may choose these paths to be piecewise linear. The lifts of these under smooth charts are still piecewise smooth and so is a finite composition of these. Thus every pair of points can be connected with a piecewise smooth path. This actually gives us a length structure on the Riemannian manifold $\mathcal{M}$. The requirements of definition 4.1 are straightforward to check.

Furthermore, we may approximate any continuous path on a smooth manifold by a piecewise smooth path. To do this one needs to cover the continuous path with chart-neighbourhoods in a useful way, use compactness of the image of the path and lift approximating piecewise smooth paths from $\mathbb{R}^n$.

This all means that the following definition is sufficient and does define a metric.

Definition 4.10. We define a path-length structure to a Riemannian manifold $\mathcal{M}$ by taking $\mathcal{C}$ to be the collection of all piecewise smooth paths on $\mathcal{M}$. The length defined in definition 4.9 makes this a path-metric structure on $\mathcal{M}$. By the previous notions a Riemannian manifold is always rectifiably connected with respect to this length structure, so the function d_ℓ is a metric.

By the basic theorems of Riemannian geometry, the metric thus given induces a topology that coincides with the original topology of the Riemannian manifold. Thus we now that *a Riemannian manifold is a length manifold.*

We noted earlier that weakly doubling metric spaces behave nicely when working with growth rate. We mention the following very nice theorem, that follows from [Kan85, Lemma 2.3, p.397]. This theorem gives us lots of examples of weakly doubling length manifolds.

Theorem 4.11. *A complete Riemannian manifold $\mathcal{M}$ whose Ricci curvature is bounded from below is weakly doubling.*

4.2.2 Lipschitz manifolds

Lipschitz manifolds are defined in a similar manner as smooth manifolds. Instead of requiring the transition mappings to be smooth, we merely require them to be bi-Lipschitz.

Let $\mathcal{M}$ be an n-dimensional manifold. We call a collection of charts of $\mathcal{M}$

$$\mathcal{A} = \{(U_\alpha, \varphi_\alpha) \mid \varphi_\alpha \colon U_\alpha \to \mathbb{R}^n, \alpha \in I\},$$

a *Lipschitz atlas* if the following two conditions hold. The sets U_α cover the manifold $\mathcal{M}$ and the *transition mappings* $\left(\varphi_\alpha \circ \varphi_\beta^{-1}\right)|_{U_\alpha \cap U_\beta}$ are bi-Lipschitz mappings.

Definition 4.12. A Lipschitz manifold is a topological manifold together with a maximal Lipschitz atlas.

To define a path-length structure for a Lipschitz manifold $\mathcal{M}$ we note that as manifolds are always paracompact by lemma 2.8. This means that we may take a locally finite refinement of the open cover of a Lipschitz manifold that consists of the neighbourhoods in the given Lipschitz atlas. This locally finite refinement gives rise to a subatlas of the original atlas. Call this locally finite atlas $\mathcal{B}$. For any path $\gamma \colon [0,1] \to \mathcal{M}$ in our manifold we can now pick for each $y \in \gamma[0,1]$ a neighbourhood U_y intersecting only finitely many elements of $\mathcal{B}$. In fact we may select U_y to be so small that it intersects the minimal amount of elements $\mathcal{B}$ possible. The image of γ is compact as a continuous image of a compact set, so $\mathcal{B}$ has a finite subcover. Let K be any finite subcover of $\mathcal{B}$. Let us define

$$\ell^K(\gamma) = \sum_{B \in K} \ell(f_B \circ \gamma),$$

where f_B is the chart mapping associated with B. Now we set

$$\ell(\gamma) = \inf(\ell^K(\gamma) \mid K \text{ a finite subcover of } \mathcal{B}).$$

This gives us a length structure of our manifold. Note that this length structure does depend on the chosen locally finite subatlas $\mathcal{B}$.

Metric structure of a Lipschitz manifold is also discussed in [LV77, Section 3.].

4.3 Binding the geometry

The results of this thesis are essentially geometric, so we need the objects examined to have a sensible geometry, at least in some sense. We will use a requirement that resembles a common requirement in this field which is that of a bounded injectivity radius. To avoid confusion our mimicking condition will be called a null-homotopy radius. We also show how a strictly positive null-homotopy radius is easily obtained by demanding bounded local geometry. We will need bounded geometry in the 'other direction' as well, so we will bind the geometry from above via the concept of a diameter bounded fundamental group. We will give both weak and strong versions of these. Both of these bounds can always be found from the concrete example of a compact length manifold.

Definition 4.13. For each $x \in \mathcal{M}$ let us denote by r^x_{nh} the supremum of those $r \geq 0$ for which any loop based on x of length at most r is homotopic to a constant path $t \mapsto x$. This number r^x_{nh} is called the *weak null-homotopy radius of* $\mathcal{M}$ *at* x.

A manifold $\mathcal{M}$ is said to have a *strictly positive weak null-homotopy radius* if for every point $x \in \mathcal{M}$ we have that $r^x_{\mathrm{nh}} > 0$.

The infimum $r_{\mathrm{nh}} := \inf\{r^x_{\mathrm{nh}} | x \in \mathcal{M}\}$ of the weak null-homotopy radii is called the *null-homotopy radius of* $\mathcal{M}$ and say that that manifold $\mathcal{M}$ *has a strictly positive null-homotopy radius* if $r_{\mathrm{nh}} > 0$.

Note that a length manifold is simply connected if and only if the null-homotopy radius $r_{\mathrm{nh}} = \infty$. Also note that a strictly positive null-homotopy radius guarantees the weak condition as well.

Theorem 4.14. *A compact length manifold* $\mathcal{M}$ *has a strictly positive null-homotopy radius.*

Proof. For every point $x \in \mathcal{M}$ we can take a chart U around x homeomorphic to $\mathbb{R}^n$ via a chart-mapping. Within this chart we pick a ball $\mathrm{B}\left(x, \varepsilon_x\right)$. Because $\mathcal{M}$ is compact, we have a Lebesgue number λ such that for all points $y \in \mathcal{M}$ that $\mathrm{B}\left(y, \lambda\right) \subset \mathrm{B}\left(x, \varepsilon_x\right)$ for some $x \in \mathcal{M}$. Thus if we take any loop $\gamma\colon [0,1] \to \mathcal{M}$ with $\ell(\gamma) < 2\lambda$ we must have that $\gamma([0,1]) \subset \mathrm{B}\left(\gamma(0), \lambda\right) \subset U$. But this means that the loop γ is null-homotopic, as it lies in a set homeomorphic to a simply connected set $\mathbb{R}^n$ via a chart.

Thus for the null-homotopy radius r_{nh} of the manifold $\mathcal{M}$ we have that $r_{\mathrm{nh}} \geq 2\lambda > 0$. □

Definition 4.15. An n-dimensional length manifold $\mathcal{M}$ has a *bounded local geometry* (sometimes abbreviated BLG) if there exists constants $L \geq 1$ and $r > 0$ such that every point $x \in \mathcal{M}$ has a neighbourhood U for which there exists an surjective L-bi-Lipschitz mapping $f\colon (U, x) \to \left(\mathrm{B}\left(0, r\right), 0\right) \subset \mathbb{R}^n$.

Note that the bounded local geometry condition arises naturally only in the category of Lipschitz-manifolds. All Lipschitz manifolds do not of course have bounded local geometry, but we do have the following result.

Theorem 4.16. *A compact Lipschitz length-manifold has bounded local geometry.*

Proof. Let $\mathcal{M}$ be a compact n-dimensional Lipschitz manifold, and $\mathcal{B}$ a locally finite subatlas that defines a length structure to $\mathcal{M}$. For any point $x \in \mathcal{M}$ we can find a neighbourhood U_x that is L_x-bi-Lipschitz -equivalent to some $B\left(0, r_x\right) \subset \mathbb{R}^n$, as we have required the transition mappings to be bi-Lipschitz maps to open subsets of $\mathbb{R}^n$. Furthermore we can choose the neighbourhoods

U_x to be balls so small that they are contained in a member of $\mathcal{B}$ and have a common lower bound r_0 to their diameter. (By using for example a Lebesgue number.)

Now the neighbourhoods U_x are L-bi-Lipschitz equivalent to unit balls of $\mathbb{R}^n$ via restrictions of the chart-mappings. (The length structure of $\mathcal{M}$ with respect to the subatlas $\mathcal{B}$ makes the chart mappings associated with $\mathcal{B}$ bi-Lipschitz mappings with a common constant.) $\square$

Theorem 4.17. *A manifold with bounded local geometry is always complete.*

Proof. Assume the BLG-constants are L and r. We pick a Cauchy sequence (x_n) from $\mathcal{M}$, and pick n_0 so large that $x_n \in \mathrm{B}\left(x_{n_0}, \frac{r}{4L}\right)$ for all $n \geq n_0$. This ball is mapped into the set $\mathrm{B}\left(0, \frac{r}{2}\right) \subset \mathbb{R}^n$ via a bi-Lipschitz map f. Thus the sequence $(f(x_n)_{n \geq n_0}$ is a Cauchy sequence and thus converges to a point y_0 in the complete space $\mathbb{R}^n$. As we picked the ball around x_{n_0} small enough, the limit lies within $\mathrm{B}\left(0, r\right)$, and we thus find a limit point $f^{-1}\{y_0\}$ to the sequence (x_n) in $\mathcal{M}$ as the mapping f is a homeomorphism. $\square$

The whole reason for us to talk about bounded local geometry is the following result.

Theorem 4.18. *A manifold $\mathcal{M}$ with bounded local geometry has a strictly positive null-homotopy radius.*

Proof. By the BLG property there exists $L \geq 1$ and $r > 0$ so that every point $x \in \mathcal{M}$ has a neighbourhood U_x and a surjective L-bi-Lipschitz map $f_x \colon U_x \to \mathrm{B}\left(0, r\right)$ with $f_x(x) = 0$.

It suffices to show that $B\left(x, \frac{r}{L}\right) \subset U_x$ for all $x \in \mathcal{M}$. Let $y \in B\left(x, \frac{r}{L}\right)$ and $\gamma \colon [0, 1] \to \mathrm{B}\left(x, \frac{r}{L}\right)$ be a path connecting the points x and y in $\mathrm{B}\left(x, \frac{r}{L}\right)$. Set

$$t_0 = \sup\left\{t \in [0, 1] \mid \gamma[0, t] \subset U_x\right\}.$$

Let us assume that $t_0 < 1$. Because $\mathrm{B}\left(x, \frac{r}{L}\right)$ is open, there exists a neighbourhood U of $\gamma(t_0)$ within $\mathrm{B}\left(x, \frac{r}{L}\right)$. By looking at a ball around x with radius less than $\frac{1}{4}\left(\frac{r}{L} - d(\gamma(x_0), x)\right)$ we see that there exists a number a such that $d(\gamma(t), x) < a < \frac{r}{L}$ for all t sufficiently close to t_0.

By the definition of t_0 and as f_x is a bi-Lipschitz homeomorphism, $f_x(\gamma(t_0)) \in \partial\,\mathrm{B}\left(0, r\right)$ and so

$$\lim_{t \to t_0-} d(f_x(\gamma(t)), \partial B(0, r)) = 0.$$

This means that for any $\varepsilon > 0$ there exists a point $s \in [0, t_0]$ such that $d(f_x(\gamma(s)), \partial B(0, r)) < \varepsilon$, which in turn implies by triangle inequality that $d(f_x(\gamma(s)), 0) > r - \varepsilon$. Moreover we can choose s to be so close to t_0 that $\gamma(s) \in U$.

Now we finally see that

$$r - \varepsilon < d(f_x(\gamma(s)), 0) = d(f_x(\gamma(s)), f_x(x)) \leq Ld(\gamma(s), x) < a < r.$$

But if we choose $\varepsilon < r - a$, this implies that $a < a$. This is a contradiction, so we must have that $t_0 = 1$.

We still need to show that $\gamma(1) = y \in U_x$. But $y \notin U_x$ would imply that $y \in \partial U_x$. Because y belongs to the open set $\mathrm{B}\left(x, \frac{r}{L}\right)$, there exists a neighbourhood A of x contained in $\mathrm{B}\left(x, \frac{r}{L}\right)$. But now as $y \in \partial U_x$, we must have $A \cap \complement U_x \neq \emptyset$. This means that we can extend our path closer to the boundary of $\mathrm{B}\left(x, \frac{r}{L}\right)$ and iterate the previous argument to create a contradiction. The claim thus holds.

□

We have already bounded the behaviour of the geometry of our manifold from below by the null-homotopy radius. Next we define a concept that bounds the behaviour from above. Combining these two criterions we get manifolds that have no geometrically interesting properties in too small scales or too far from some fixed area. We again give both a weak and a strong version of the requirement.

Definition 4.19. We say that a length manifold $\mathcal{M}$ has a *weakly diameter bounded fundamental group* if for every point $x \in \mathcal{M}$ there exists a constant $K_{\Pi}^x \in \mathbb{R}$ so that for any homotopy class $[\gamma] \in \Pi_1(\mathcal{M}, x)$ contains an element α with $d(|\alpha|) \leq K_{\Pi}^x$.

If a constant K_{Π} can be chosen globally, we say that the fundamental group of $\mathcal{M}$ is *diameter bounded.*

The idea of a strictly positive (weak) null-homotopy radius and an (weakly) diameter bounded fundamental group is to abstract those properties of a compact manifold that guarantee nice properties for the fundamental group. From the homotopic point of view we have no problem working with, for example, the cylinder $\mathbb{R} \times S^1$, but restricting ourselves to compact manifolds would rule out such spaces. Furthermore, our main results can be enhanced by using the combined growth of the original manifold and its fundamental group. In these cases we need (among other constraints) the strong versions of bounds on our geometry to get the structure of our manifold to be homogeneous enough. This result of combined growth yields something similar to the Varopoulos type result, but the general case where we allow the base space to grow itself is very natural and much more interesting in the sense of growth.

The following theorem and its corollary are trivial although useful for some of the corollaries of our main results.

Theorem 4.20. *A bounded length manifold has diameter bounded fundamental group.*

Corollary 4.21. *A compact length manifold has diameter bounded fundamental group.*

Theorem 4.22. *Let $\mathcal{M}$ be a length manifold. Suppose there exists a bounded set A such that any loop $\gamma\colon [0,1] \to \mathcal{M}$ with $\gamma(0) \in A$ is homotopic to a loop that lies within the set A. Then $\mathcal{M}$ has a weakly diameter bounded fundamental group.*

Proof. This follows easily as we now clearly have $d(|\alpha|) \leq d(A)d(x_0, A) + 1$ for any loop α with $\alpha(0) = x_0$. □

4.4 Ascended structures of the universal cover

We now turn again to the structure of covers of a manifold. We wish to check which properties defined so far can be lifted to the universal cover, and in what ways can we extract more specific information about the universal cover. We first show that some of the path-metric concepts like the bounds on the geometry ascend to the (universal) cover.

Corollary 4.23. *Suppose $\mathcal{M}$ is a length-manifold and $\widehat{\mathcal{M}}$ its cover. Then:*

1.) If $\mathcal{M}$ has a strictly positive (weak) null-homotopy radius, so does $\widehat{\mathcal{M}}$. Moreover, denote the (weak) injectivity radii of $\mathcal{M}$ and $\widehat{\mathcal{M}}$ by r and r', respectively. Then $r \leq r'$. Also the null-homotopy radius of the universal cover is always ∞.

2.) If $\mathcal{M}$ has a bounded local geometry, so does $\widehat{\mathcal{M}}$.

3.) The manifold $\mathcal{M}$ is complete if and only if $\widehat{\mathcal{M}}$ is complete.

Remark 4.24. we already know that the universal cover of a manifold is a manifold itself, so all the claims are well-defined.

Proof. We prove the claims one at a time. We shall receive these 'induced' by the covering map.

1.) The fact that the universal cover has strictly positive null-homotopy radius follows immediately from the previously stated fact that the universal cover of a manifold is simply connected and thus $r_{\mathrm{nh}} = \infty$. Thus we will have our main interest in this proof in the 'non-universal' covers of $\mathcal{M}$.

Let $r^x_{\mathrm{nh}} > 0$ be the null-homotopy radius of $\mathcal{M}$. We immediately note that whenever we take a loop α in our cover with length less than r^x_{nh}, the loop $p_{\mathcal{M}} \circ \alpha$ in $\mathcal{M}$ has also length less than r^x_{nh} and is thus null-homotopic. Homotopies lift to the cover via the covering map by basic results of covers, so the original loop is also null-homotopic. Thus the cover $\widehat{\mathcal{M}}$ has null-homotopy radius at least r^x_{nh} at every point $y \in p^{-1}_{\mathcal{M}}\{x\}$. The 'non-weak' case follows from this.

We do find it fascinating that if we create an ordering on the covers of a manifold based on whether or not they are covers of each other, this ordering respects the injectivity radii. See corollary 4.25 for a more specific formulation.

2.) Let the 'BLG-constants' of $\mathcal{M}$ be L and r. By the definition of the length-metric on any cover, the covering map comes a local isometry. As the BLG-neighbourhoods of any point $x \in \widehat{\mathcal{M}}$ are simply connected, they are all lifted as disjoint BLG-neighbourhoods of points $p^{-1}_{\mathcal{M}}\{x\}$. This proves the claim.

3.) The claim follows as we have defined the metric in any cover such that the covering map is a local isometry. ☐

Corollary 4.25. *Take all the covers of a manifold $\mathcal{M}$ and order them by setting $(\widehat{\mathcal{M}}_1, p^1_{\mathcal{M}}) \leq (\widehat{\mathcal{M}}_2, p^2_{\mathcal{M}})$ if there exists a covering map $p\colon \widehat{\mathcal{M}}_2 \to \widehat{\mathcal{M}}_1$ such that $p^1_{\widehat{\mathcal{M}}} \circ p = p^2_{\widehat{\mathcal{M}}}$. Now if two covers of $\mathcal{M}$ are in some order, their injectivity radii are in the same order.*

Proof. This follows from the proof of part 2) of the previous theorem. ☐

Remark 4.26. There is a very natural well-known map that embeds the fundamental group of a manifold into its universal cover. As the lifts of two loops in a manifold have the same endpoint if and only if the original loops are homotopic, we may define

$$\varphi^x_{\mathcal{M}}\colon \Pi_1(\mathcal{M}, x) \to (\widetilde{\mathcal{M}}, \widetilde{x}), \quad \varphi^x_{\mathcal{M}}([\gamma]) = \widetilde{\gamma}(1).$$

for any $x \in \mathcal{M}$, $\widetilde{x} \in p^{-1}\{x\}$. (Here the lifted paths are understood to start from $\widetilde{x}$.)

The mapping $\varphi_{\mathcal{M}}^{x}$ is always injective. Next we see that a strictly positive null-homotopy radius r_{nh} of a manifold $\mathcal{M}$ ascends into a bound for the discreteness of the image of $\varphi_{\mathcal{M}}$. More precisely we state the following theorem, which would give us justification to call our null-homotopy radius as 'injectivity radius' instead.

Theorem 4.27. *Let $\mathcal{M}$ be a manifold with strictly positive weak null-homotopy radius r_{nh}^{x} at $x \in \mathcal{M}$. Then for any two disjoint $x, y \in \operatorname{Im} \varphi_{\mathcal{M}}$ we have $d(x,y) \geq r_{nh}$.*

Proof. The claim follows immediately from the definition of weak null-homotopy radius and from the fact that any path between two distinct points of $\operatorname{Im} \varphi_{\mathcal{M}}^{x}$ represents a nonzero element of $\Pi_1(\mathcal{M}, x)$. □

Note that a similar map could be defined from $\Pi_1(\mathcal{M}, x)$ to any cover of $\mathcal{M}$. The previous theorem would not hold in its current form, but we would have a weaker result stating that the distances between image points are zero or at least r_{nh}.

We are now ready to improve the result of theorem 2.11 for length manifolds with strictly positive null-homotopy radius and an diameter bounded fundamental group. The following proof is adapted version of the proof of a similar theorem, namely the first part of [Gro99, Proposition 3.22., p.90]. However in the book of Gromov the result is proven only for compact manifolds.

Theorem 4.28. *Let $(\mathcal{M}, x_0)$ be a complete length manifold with strictly positive weak null-homotopy radius at x_0 and a weakly diameter bounded fundamental group at x_0. Then the fundamental group $\Pi_1(\mathcal{M}, x_0)$ of $(\mathcal{M}, x_0)$ is finitely generated.*

If in addition $r_{nh} > 0$, the spanning set can be chosen to be a finite subset of

$$G = \{[\alpha * \beta * \gamma] \in \Pi_1(\mathcal{M}, x_0) \mid \alpha \colon x_0 \underset{g}{\rightsquigarrow} x, \beta \colon x \underset{g}{\rightsquigarrow} y, \gamma \colon y \underset{g}{\rightsquigarrow} x_0, x, y \in \mathcal{M}\}.$$

Proof. As we assumed the manifold to have an weakly diameter bounded fundamental group, we know that there exists a constant $K := K_{\Pi}^{x_0}$ such that all homotopy classes in $\Pi_1(\mathcal{M}, x_0)$ contain a loop with diameter less than K. Let $\varepsilon_1 > 0$ be arbitrary. For each element $[\gamma]$ of the fundamental group $\Pi_1(\mathcal{M}, x_0)$ we choose a rectifiable representative loop $\gamma \colon [0,1] \to \mathcal{M}$ with $d(|\gamma|) \leq K$. Call this set of representatives $P_1(\mathcal{M}, x_0)$. For each path γ we take a partition

$$0 = a_0^{\gamma} < a_1^{\gamma} < \cdots < a_n^{\gamma} = 1$$

of the domain interval such that $\ell(\gamma_{[a_i^{\gamma}, a_{i+1}^{\gamma}]}) < \varepsilon_1$ for all $i = 0, \ldots, n-1$.

Next for all $\gamma \in P_1(\mathcal{M}, x_0)$ we take geodesics

$$c_j^{\gamma} \colon x_0 \underset{g}{\rightsquigarrow} \gamma(a_j^{\gamma}), \quad \ell(c_j^{\gamma}) = d(x_0, \gamma(a_j^{\gamma})),$$

where $j = 0, \ldots n$. Now each path γ can be represented as a product of loops of the form $c_i^{\gamma} \gamma_{[a_i, a_{i+1}]} \overleftarrow{c_{i+1}^{\gamma}}$:

$$\gamma = \gamma_{[a_0,a_1]} \cdots \gamma_{[a_{n-1},a_n]} \sim \left(c_0^{\gamma} \gamma_{[a_0,a_1]} \overleftarrow{c_1^{\gamma}}\right) \left(c_1^{\gamma} \gamma_{[a_1,a_2]} \overleftarrow{c_2^{\gamma}}\right) \cdots \left(c_{n-1}^{\gamma} \gamma_{[a_{n-1},a_n]} \overleftarrow{c_n^{\gamma}}\right).$$

By the definition of $P_1(\mathcal{M}, x_0)$ we have that $d(x_0, \gamma(a_i)) < K$ for all points a_i. We now see that

$$\begin{aligned}\ell(c_i^\gamma \gamma_{[a_i,a_{i+1}]} \overleftarrow{c_{i+1}^\gamma}) &= \ell(c_i^\gamma) + \ell(\gamma_{[a_i,a_{i+1}]}) + \ell(\overleftarrow{c_{i+1}^\gamma}) \\ &\leq d\left(x_0, \gamma(a_i)\right) + \varepsilon_1 + d\left(x_0, \gamma(a_{i+1})\right) \\ &\leq 2K + \varepsilon_1 < \infty.\end{aligned}$$

This means that all the representatives can be expressed as a product of loops with length less than R, where $R = 2K + \varepsilon_1$ is independent of any choice of homotopy classes or their representatives. This means that all elements of the fundamental group can be expressed as a product of homotopy classes of loops containing a loop with length less than R. All we need to show now is that there can be only finitely many such classes. For this we shall need the strictly positive weak null-homotopy radius at x_0.

Let us pick $\widetilde{x_0} \in p_{\mathcal{M}}^{-1}\{x_0\}$. We defined the path-length structure in the universal cover by demanding that the covering map preserves path-lengths. This means that the loops with length less than R in $\mathcal{M}$ are lifted to paths in $\widetilde{\mathcal{M}}$ also with length less than R. Thus it will suffice to show that the following set, which is basically the image of the loops with bounded length under an injective map,

$$B_* = \operatorname{Im} \varphi_{\mathcal{M}} \cap B\big(\widetilde{x_0}, R\big)$$

is finite. But this is now easy, as by theorem 4.27 there is a lower bound to the distances between points in B_* given by the weak null-homotopy radius $r := r_{\mathrm{nh}}^{x_0}$, so it is a subset of an r-net in a bounded subset of a complete length manifold, so by corollary 4.6 it must by finite and this proves the first claim.

Suppose now that in addition $r_{\mathrm{nh}} > 0$. By rewriting the beginning of the proof we may take the partition $a_1 < \ldots < a_n$ to be so small that the restrictions $\gamma|_{[}a_j, a_{j+1}]$ are homotopic to geodesics. Substituting these in place of the paths γ_i we get the second claim. $\square$

We have the following corollary that is amusing, but useless for us as homotopy equivalences do not have to preserve any geometric structure.

Theorem 4.29. *Assume that $\mathcal{M}$ is a manifold homotopy equivalent to a complete path-length manifold that satisfies the criterions of theorem 4.28. Then the fundamental group $\Pi_1(\mathcal{M})$ of the manifold $\mathcal{M}$ is finitely generated.*

Proof. The claim follows as a homotopy equivalence induces an isomorphism between fundamental groups of topological spaces, and the property of a group being finitely generated is preserved under isomorphisms. $\square$

Remark 4.30. In theorem 4.23 we gave results for the ascension of the null-homotopy radius, but nothing for diameter bounded fundamental group. This is because the universal cover is always simply connected and has thus a diameter bounded fundamental group, but any other covers do not need to have this property as we shall now see.

We like to think that going 'up' along covering spaces always 'opens up' the topological space in question. This means that it is quite natural that as is the case with boundedness, an arbitrary cover of a manifold with a diameter bounded fundamental group rarely has a diameter bounded fundamental group itself. This can be seen by studying any compact manifold $\mathcal{M}$ that has the free group of two elements as a subgroup of its fundamental group. We know that the the commutator subgroup H of the free group of two elements, is not finitely generated. Especially there H is a non finitely generated subgroup of $\Pi_1(\mathcal{M})$. By a interesting result of covering spaces, whenever we have a topological space X which has a universal cover, we find for every subgroup $G \leq \Pi_1(X)$ a cover $\hat{X}$ of X such that $\Pi_1 \hat{X} = G$, and $\hat{X} = \widetilde{X} \big/ G$. Especially there exists a cover Y of the manifold $\mathcal{M}$ that has the non-finitely generated group H as a fundamental group. This cover is by parts 1) and 2) of theorem 4.23 a path-length manifold with a strictly positive null-homotopy radius as a cover of a compact manifold is complete. But by theorem 4.28 it cannot have a diameter bounded fundamental group, because then $\Pi_1(Y)$ would be finitely generated.

We now turn our attention to getting more qualitative coarse information from our universal cover. It is one of the basic results of coarse geometry that the fundamental group of a compact manifold is coarsely quasi-isometric to the universal cover of the manifold in question. The result does not hold for general length manifolds even with strictly positive null-homotopy radii and diameter bounded fundamental groups, one needs just to look at the case of the cylinder $\mathbb{R} \times S^1$. But by studying this example and comparing it to the case of torus, whose covering space the cylinder is, one notes that if we lift the fundamental group of the torus to the cylinder, and the fundamental group of the cylinder to $\mathbb{R}^2$, we 'lose' something. This something is the part of the fundamental group of the torus that is lifted to the 'whole height' of the cylinder, and this 'height' is what we need to take into consideration in the general case. More specifically, this height shows itself in the coarse theory in the fact that the cylinder has non-zero growth rate. Thus half of the fundamental group of the torus goes to the fundamental group of the cylinder, whose appearance on the cylinder is bounded and without growth rate, and half of it goes to the growth rate of the cylinder. To take these both to consideration we must in general start to construct our coarse quasi-isometry by taking as the domain not just the fundamental group, but the product $\mathcal{M} \times \Pi_1(\mathcal{M})$. To get the wanted mapping, we look for the essential part of the standard coarse quasi-isometry $\varphi_{\mathcal{M}}$ from the fundamental group of a compact manifold to the universal cover, and mimic it to get the other 'half' of the mapping. The result we now start to prove is quite strong, so we also need to require some further constraints on the geometry of the manifolds in question.

We wish to imitate the idea behind the mapping defined in remark 4.26. The mapping itself is good, as it already gives us half of what we want, but we want to expand the domain in question. Because lifting of paths is easy under covering maps we want to represent the points in our manifold by paths. Thus we will basically fix a point and choose for each point in the manifold a geodesic connecting it to the fixed point. Now the manifold *and* fundamental group appear as paths starting from a given point, so we may use the function $\varphi_{\mathcal{M}}$ with an extended domain. We need first a few lemmas to further bind together the geometric structures of a manifold and its fundamental group.

Suppose $\mathcal{M}$ is a length manifold with strictly positive weak null-homotopy radius r^x_{nh} at x. As $r^x_{\mathrm{nh}} > 0$, we can define a 'norm'

$$\ell^* \colon \Pi_1(\mathcal{M}, x) \to \mathbb{R}_+, \quad \ell^*([\omega]) = \inf\{\ell(\gamma) \mid \gamma \in [\omega]\},$$

and via this norm a metric

$$d^x_\ell \colon \Pi_1(\mathcal{M}, x) \to \mathbb{R}_+, \quad d^x_\ell([\alpha], [\beta]) = \ell^*([\overleftarrow{\alpha}\beta]).$$

Definition 4.31. We say that a complete length manifold $\mathcal{M}$ with strictly positive null-homotopy radius and a diameter bound fundamental group is *geometrically homogeneous* if the two following conditions hold.

(1) There exists a global constant L such that for any $x \in \mathcal{M}$ the metrics d^x_ℓ and d_S of $\Pi_1(\mathcal{M}, x)$ are L-bi-Lipschitz equivalent.

(2) For any fixed $x_0 \in \mathcal{M}$, the set

$$G := \{[\alpha * \beta * \gamma] \in \Pi_1(\mathcal{M}, x_0) \mid \alpha \colon x_0 \underset{g}{\curvearrowright} x, \beta \colon x \underset{g}{\curvearrowright} y, \gamma \colon y \underset{g}{\curvearrowright} x_0, x, y \in \mathcal{M}\}$$

of geodesic triangles is finite.

If there exists a loop with smallest length in $[\omega] \in \Pi_1(X, x)$, we denote by $\omega^* \in [\omega]$ any loop such that $\ell(\omega^*) = \ell^*([\omega])$. If we have a weak diameter bound K^x_Π for our fundamental group and $[\omega] \in \Pi_1(X, x)$, we denote by $\omega_* \in [\omega]$ any loop such that $d(|\omega_*|) \leq K^x_\Pi$.

Let $\mathcal{M}$ be a complete length manifold, $x_0 \in \mathcal{M}$ a fixed point and $\tilde{x}_0 \in p^{-1}_{\mathcal{M}}\{x_0\} \subset \widetilde{\mathcal{M}}$. We define first

$$\vartheta \colon \mathcal{M} \to \mathcal{C}([0,1]), \quad \vartheta(x) \colon x_0 \underset{g}{\curvearrowright} x.$$

Note that geodesics on a length manifold are rarely unique, but here we just pick *a* geodesic between x_0 and x.

Furthermore we define

$$\Theta \colon \mathcal{M} \to \widetilde{\mathcal{M}}, \quad \Theta(x) = \widetilde{\vartheta}(x)(1),$$

where $\widetilde{\vartheta}(x)$ is the lift of $\vartheta(x)$ starting from $\tilde{x}_0$.

Lemma 4.32. *Suppose $x, y \in \mathcal{M}$ and $[\omega] \in \Pi_1(\mathcal{M}, x_0)$. Then*

$$\tilde{d}\left((\widetilde{\omega} * \widetilde{\vartheta}(x))(1), (\widetilde{\omega} * \widetilde{\vartheta}(y))(1)\right) = \tilde{d}\left(\widetilde{\vartheta}(x)(1), \widetilde{\vartheta}(y)(1)\right) = \tilde{d}\left(\Theta(x), \Theta(y)\right).$$

Proof. The second equality is just the definition of Θ, so we concentrate on the first one.

Let $\beta \colon x \underset{g}{\curvearrowright} y$. Now there are two lifts of β,

$$\widetilde{\beta}_1 \colon (\widetilde{\omega} * \widetilde{\vartheta}(x))(1) \underset{g}{\curvearrowright} (\widetilde{\omega} * \widetilde{\vartheta}(y))(1) \text{ and}$$
$$\widetilde{\beta}_2 \colon \widetilde{\vartheta}(x)(1) \underset{g}{\curvearrowright} \widetilde{\vartheta}(y)(1).$$

by definition of $\tilde{d}$, we must have $\widetilde{\ell}(\widetilde{\beta}_1) = \ell(\beta) = \widetilde{\ell}(\widetilde{\beta}_2)$, and this proves the claim. □

We now begin to construct our coarse quasi-isometry.

Lemma 4.33. *Let $\mathcal{M}$ be a geometrically homogeneous length manifold. There exists a constant $D > 0$ such that*

$$d(x,y) - D \leq \tilde{d}(\Theta(x), \Theta(y)) \leq d(x,y) + D.$$

Proof. Let $x, y \in \mathcal{M}$ and let $\beta \colon x \underset{G}{\curvearrowright} y$. We note that now the composition

$$\sigma := \Theta(x) * \beta * \Theta^{-1}(y)$$

is a geodesic triangle, so $\sigma \in G$. Moreover we see that as the loops $\sigma * \vartheta(y)$ and $\vartheta(x) * \beta$ are homotopic, we must have

$$\left(\widetilde{\sigma} * \widetilde{\vartheta}(y)\right)(1) = \left(\widetilde{\vartheta}(x) * \widetilde{\beta}\right)(1).$$

By requirement (1) of definition 4.31 we see that $\ell^*([\sigma]) \leq L\|[\sigma]\|_S$. As $[\sigma] \in G$ and G is finite by requirement (2) of definition 4.31, we also have that

$$\ell^*([\sigma]) \leq L\|[\sigma]\|_S \leq L \max_{g \in G} \|g\|_S := C < \infty.$$

Now by using triangle inequality we see that

$$\begin{aligned}
d(\Theta(x), \Theta(y)) &\leq d(\Theta(x), (\widetilde{\vartheta}(x) * \widetilde{\beta})(1)) + d((\widetilde{\vartheta}(x) * \widetilde{\beta})(1), \Theta(y)) \\
&= \ell(\widetilde{\beta}) + \ell^*([\sigma]) \\
&= \ell(\beta) + \ell^*([\sigma]) \\
&\leq d_\ell(x,y) + D
\end{aligned}$$

and

$$\begin{aligned}
d(\Theta(x), \Theta(y)) &\geq d(\Theta(x), (\widetilde{\vartheta}(x) * \widetilde{\beta})(1)) - d((\widetilde{\vartheta}(x) * \widetilde{\beta})(1), \Theta(y)) \\
&= \ell(\beta) - \ell(\widetilde{\sigma}) \\
&\geq d_\ell(x,y) - D.
\end{aligned}$$

This proves the claim.

∎

Lemma 4.34. *Let $\mathcal{M}$ be a geometrically homogenous length manifold and L be a constant such that the metrics d_S and d_ℓ^x of $\Pi_1(\mathcal{M}, x)$ are L-bi-Lipschitz equivalent for any $x \in \mathcal{M}$. Suppose $K \in \,]0, L^{-1}[$. Furthermore let $x, y \in \mathcal{M}$ and $[\sigma], [\gamma] \in \Pi_1(\mathcal{M}, x_0)$ for some fixed $x_0 \in \mathcal{M}$. Then the following hold.*

(i) If $\|[\overleftarrow{\sigma} * \gamma]\|_S \leq K d_\ell(x,y)$ then

$$\tilde{d}(\Theta(x), (\widetilde{\overleftarrow{\sigma}} * \widetilde{\gamma} * \widetilde{\vartheta}(y))(1)) \geq (1 - KL) d_\ell(x,y) - C.$$

(ii) If $\|[\overleftarrow{\sigma} * \gamma]\|_S \geq K d_\ell(x,y)$ then

$$\tilde{d}(\Theta(x), (\widetilde{\overleftarrow{\sigma}} * \widetilde{\gamma} * \widetilde{\vartheta}(y))(1)) \geq (L^{-1} - K)\, \|[\overleftarrow{\sigma} * \gamma]\|_S - D.$$

The constant D is the one used in lemma 4.33. Note that by the choice of K both coefficients $(1 - KL)$ and $(L^{-1} - K)$ are strictly positive.

Proof. We denote $\alpha := \overleftarrow{\sigma} * \gamma$.

To prove the first inequality we use triangle inequality, condition (1) of the definition of a geometrically homogenous length manifold and our assumption:

$$\begin{aligned}
\tilde{d}(\Theta(x), \Theta(y)) &\leq \tilde{d}(\Theta(x), (\widetilde{\alpha} * \widetilde{\vartheta}(y))(1)) + \tilde{d}((\widetilde{\alpha} * \widetilde{\vartheta}(y))(1), \Theta(y)) \\
&= \tilde{d}(\Theta(x), (\widetilde{\alpha} * \widetilde{\vartheta}(y))(1)) + \ell_y^*([\alpha]) \\
&\leq \tilde{d}(\Theta(x), (\widetilde{\alpha} * \widetilde{\vartheta}(y))(1)) + L \left\|[\alpha]\right\|_S \\
&\leq \tilde{d}(\Theta(x), (\widetilde{\alpha} * \widetilde{\vartheta}(y))(1)) + LK d_\ell(x, y).
\end{aligned}$$

So by substracting $LK d_\ell(x, y)$ from this and noting that by lemma 4.33

$$\tilde{d}(\Theta(x), \Theta(y)) \geq d_\ell(x, y) - D$$

we see that

$$\tilde{d}(\Theta(x), (\widetilde{\alpha} * \widetilde{\vartheta}(y))(1)) \geq (1 - KL) d_\ell(x, y) - D$$

and this proves the first claim.

For the second claim we use condition (1) of the definition of a geometrically homogenous length manifold, lemma 4.32, triangle inequality, lemma 4.33 and our assumption.

$$\begin{aligned}
L^{-1} \left\|[\alpha]\right\|_S &\leq \ell^*([\alpha]) \\
&= \tilde{d}(\widetilde{x}_0, \widetilde{\alpha}(1)) \\
&= \tilde{d}(\Theta(y), (\widetilde{\alpha} * \widetilde{\vartheta}(y))(1)) \\
&\leq \tilde{d}(\Theta(x), (\widetilde{\alpha} * \widetilde{\vartheta}(y))(1)) + \tilde{d}(\Theta(x), \Theta(y)) \\
&\leq \tilde{d}(\Theta(x), (\widetilde{\alpha} * \widetilde{\vartheta}(y))(1)) + d_\ell(x, y) + D \\
&\leq \tilde{d}(\Theta(x), (\widetilde{\alpha} * \widetilde{\vartheta}(y))(1)) + K \left\|[\alpha]\right\|_S + D,
\end{aligned}$$

So we see that

$$\tilde{d}(\Theta(x), (\widetilde{\alpha} * \widetilde{\vartheta}(y))(1)) \geq (L^{-1} - K) \left\|[\alpha]\right\|_S - D.$$

Thus the second claim is also proved.

□

Theorem 4.35. *Suppose $\mathcal{M}$ is a path length manifold with strictly positive null-homotopy radius and a diameter bounded fundamental group. Then there exists a coarse quasi-isometry*

$$\Psi \colon \mathcal{M} \times \Pi_1(\mathcal{M}) \to \widetilde{\mathcal{M}},$$

when the metric in the product is defined as

$$d((x, [\alpha]), (y, [\beta])) = d_\ell(x, y) + d_S([\alpha], [\beta])$$

Proof. Let $x_0 \in \mathcal{M}$ and $\tilde{x}_0 \in \widetilde{\mathcal{M}}$.

We define

$$\Psi \colon (\mathcal{M}, x_0) \times \Pi_1(\mathcal{M}, x_0) \to \widetilde{\mathcal{M}} \quad \Psi(x, [\sigma]) = \left(\widetilde{\sigma} * \widetilde{\vartheta}(x)\right)(1).$$

We wish to show that this mapping is a coarse quasi-isometry.

The image of Ψ is full in $\widetilde{\mathcal{M}}$, as we will see by showing that the mapping Ψ is in fact surjective. For any $x \in \widetilde{\mathcal{M}}$ we take a path $\gamma\colon x_0 \curvearrowright x$. Now

$$\gamma \sim \gamma * \widetilde{\vartheta}(p_{\mathcal{M}}(x)) * \overleftarrow{\widetilde{\vartheta}(p_{\mathcal{M}}(x))}.$$

Especially

$$x = \Psi(x, [\gamma * \widetilde{\vartheta}(p_{\mathcal{M}}(x))]),$$

so the mapping Ψ is surjective.

We first see that by using the triangle inequality, the definition of Ψ, path-conjugation lemma 4.32 and the definition of a geometrically homogeneous manifold we have

$$\begin{aligned}
\tilde{d}(\Psi(x,[\sigma]),\Psi(y,[\gamma])) &\leq \tilde{d}\,(\Psi(x,[\sigma]),\Psi(y,[\sigma])) + \tilde{d}\,(\Psi(x,[\sigma]),\Psi(y,[\gamma])) \\
&\quad + \tilde{d}\,(\Psi(x,[\gamma]),\Psi(y,[\gamma])) \\
&\leq \tilde{d}\left((\widetilde{\sigma} * \widetilde{\vartheta}(x))(1), (\widetilde{\sigma} * \widetilde{\vartheta}(y))(1)\right) + \tilde{d}\left((\widetilde{\sigma} * \widetilde{\vartheta}(x))(1), (\widetilde{\gamma} * \widetilde{\vartheta}(y))(1)\right) \\
&\quad + \tilde{d}\left((\widetilde{\gamma} * \widetilde{\vartheta}(x))(1), (\widetilde{\gamma} * \widetilde{\vartheta}(y))(1)\right) \\
&\leq 2\tilde{d}\left(\widetilde{\vartheta}(x)(1), \widetilde{\vartheta}(y)(1)\right) + \tilde{d}\left((\widetilde{\vartheta}(x))(1), (\widetilde{\overleftarrow{\sigma}} * \widetilde{\gamma} * \widetilde{\vartheta}(y))(1)\right) \\
&= 2\tilde{d}(\Theta(x),\Theta(y)) + \ell^*([\overleftarrow{\sigma} * \gamma]) \\
&\leq (2d(x,y)_\ell + D) + L\,\|[\overleftarrow{\sigma} * \gamma]\|_S \\
&= 2d_\ell(x,y) + L d_S([\sigma],[\gamma]) + D \\
&\leq C d((x,[\sigma]),(y,[\gamma])) + D.
\end{aligned}$$

For the other inequality we use lemma 4.34 with constant $K = (2L)^{-1}$. (This means that $(1 - KL) = 1/2$ and $(L^{-1} - K) = L^{-1}$.)

Assume first that $\|[\overleftarrow{\sigma} * \gamma\|_S \leq (2L)^{-1} d_\ell(x,y)$. Now by the definition of Ψ and lemma 4.33 we see that

$$\begin{aligned}
\tilde{d}(\Psi(x,[\sigma]),\Psi(y,[\gamma])) &= \tilde{d}\left((\widetilde{\sigma} * \widetilde{\vartheta}(x))(1), (\widetilde{\gamma} * \widetilde{\vartheta}(y))(1)\right) \\
&\geq \frac{1}{2} d_\ell(x,y) - D \\
&= \frac{1}{4} d_\ell(x,y) + \frac{1}{4} d_\ell(x,y) - D \\
&\geq \frac{1}{4} d_\ell(x,y) + \frac{1}{L}\,\|[\overleftarrow{\sigma} * \gamma]\|_S - D \\
&\geq C' d\,((x,[\sigma]),(y,[\gamma])) - D.
\end{aligned}$$

In the case $\|[\overleftarrow{\sigma} * \gamma\|_S \geq (2L)^{-1} d_\ell(x,y)$. we have by the definition of Ψ, definition of geometrically homogeneous manifold and lemma 4.33 that

$$\begin{aligned}
\tilde{d}(\Psi(x,[\sigma]),\Psi(y,[\gamma])) &= \tilde{d}\left((\widetilde{\sigma} * \widetilde{\vartheta}(x))(1), (\widetilde{\gamma} * \widetilde{\vartheta}(y))(1)\right) \\
&= \tilde{d}\left((\widetilde{\vartheta}(x))(1), (\widetilde{\overleftarrow{\sigma}} * \widetilde{\gamma} * \widetilde{\vartheta}(y))(1)\right) \\
&L^{-1}\,\|[\overleftarrow{\sigma} * \gamma]\|_S - D \\
&= (2L)^{-1}\,\|[\overleftarrow{\sigma} * \gamma]\|_S + (2L)^{-1}\,\|[\overleftarrow{\sigma} * \gamma]\|_S - D \\
&\geq C'' d\,((x,[\sigma]),(y,[\gamma])) - D.
\end{aligned}$$

Combining these two we see that always

$$\tilde{d}(\Psi(x,[\sigma]),\Psi(y,[\gamma])) \geq C^{-1} d\left((x,[\sigma]),(y,[\gamma])\right) - D.$$

$\square$

In the cases where either $\mathcal{M}$ or $\Pi_1(\mathcal{M})$ is a point in the sense of coarse quasi-isometry, we have the following corollaries that follow basically from corollary 3.24 that states that product with a bounded metric space does not affect the coarse equivalence class.

Corollary 4.36. *Let $\mathcal{M}$ be a bounded length manifold with a strictly positive null-homotopy radius. Then the metric spaces $\widetilde{\mathcal{M}}$ and $\Pi_1(\mathcal{M})$ are coarsely quasi-isometric.*

Corollary 4.37. *Let $\mathcal{M}$ be a compact length manifold. Then the metric spaces $\widetilde{\mathcal{M}}$ and $\Pi_1(\mathcal{M})$ are coarsely quasi-isometric.*

Corollary 4.38. *Let $\mathcal{M}$ be a manifold with a strictly positive null-homotopy radius and a finite fundamental group. Then the metric spaces $\widetilde{\mathcal{M}}$ and $\mathcal{M}$ are coarsely quasi-isometric.*

The following corollary follows from basic results of covering spaces, but we state it here as a fun corollary to our coolest result.

Corollary 4.39. *Let $\mathcal{M}$ be a simply connected manifold. Then the metric spaces $\widetilde{\mathcal{M}}$ and $\mathcal{M}$ are coarsely quasi-isometric.*

5 BLD mappings

We have now defined the basic concepts we need and learned enough from their structure to start working towards our main results. Especially we have enough vocabulary to define BLD mappings and formulate their most important basic properties. After this we are ready to apply our results.

5.1 Definition of a BLD-mapping

We begin by introducing our main object of study; a BLD-mapping between manifolds. As was mentioned in the introduction of this thesis, BLD mappings can be seen to be a continuous analogue to the quasiregular mappings of analysis. As we are soon able to show, BLD mappings are Lipschitz quotient when the domain is complete. But they also have the stronger property that the lift of a BLD mapping is still a BLD mapping, a property not shared by coarse Lipschitz quotient or Lipschitz quotient mappings. This allows BLD mappings to be able to 'see' more aspects of the growth rate of their domain than Lipschitz quotient mappings.

Definition 5.1. Let $\mathcal{M}$ and $\mathcal{N}$ be two n-dimensional length manifolds. An open, discrete, continuous mapping

$$f\colon \mathcal{M} \to \mathcal{N}$$

is called a mapping of *Bounded Length Distortion*, or shortly a *BLD-mapping* if the following path-length criterion is satisfied. Whenever γ is a rectifiable path in $\mathcal{M}$, then the path $f \circ \gamma$ is also rectifiable and we have that

$$\frac{1}{L}\ell\,(f \circ \gamma) \leq \ell\,(\gamma) \leq L\ell\,(f \circ \gamma) \tag{2}$$

for some constant $L \in \mathbb{R}$, $L > 1$.

If we wish to emphasize the constant L in the BLD-criterion, we may call the mapping an L-BLD mapping.

Note that BLD mappings are defined only between manifolds of the same dimension.

Remark 5.2. A BLD-mapping need not to be a local bi-Lipschitz map or even a local homeomorphism. A classical example is to take the mapping $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, which can be in polar coordinates defined as $(r, \vartheta) \mapsto (r, 2\vartheta)$. This mapping can be seen to be a 2-BLD mapping, but it is not a local homeomorphism at the origin. We shall later study the set of points at which a BLD map fails to be a local homeomorphism. What we do have, however, is the following result.

Lemma 5.3. *An L-BLD mapping $f\colon \mathcal{M} \to \mathcal{N}$ between length manifolds is an L-Lipschitz map.*

Proof. Let $x, y \in \mathcal{M}$ and $\varepsilon > 0$. We can pick a rectifiable path $\gamma\colon \mathcal{M} \to \mathcal{N}$ connecting these points with $\ell(\gamma) \leq d(x, y) + \varepsilon$. Now we see that the path $f \circ \gamma$ connects the points $f(x)$ and $f(y)$. Moreover we have that

$$d(f(x), f(y)) \leq \ell(f \circ \gamma) \leq L\ell(\gamma) \leq L(d(x, y) + \varepsilon).$$

As this holds for any $\varepsilon > 0$, the mapping is L-Lipschitz. $\square$

Our main quest is to give restraints to the existence of BLD mappings between manifolds. Our maybe most important concrete case is that of the existence of BLD mappings from $\mathbb{R}^n$ to a compact manifold. The following example shows that this question comes void if we remove the requirement of openness from the definition of a BLD mapping.

Example 5.4. Let us define a mapping $f\colon \mathbb{R}^2 \to [0,1] \times [0,1]$, by setting

$$f(x,y) = ((-1)^n(x-n), (-1)^m(x-m)),$$

where $m, n \in \mathbb{Z}$ are such that $x \in [m, m+1[, y \in [n, n+1[$. The map is generated by folding the plane, as illustrated in figure 8. This mapping preserves path-

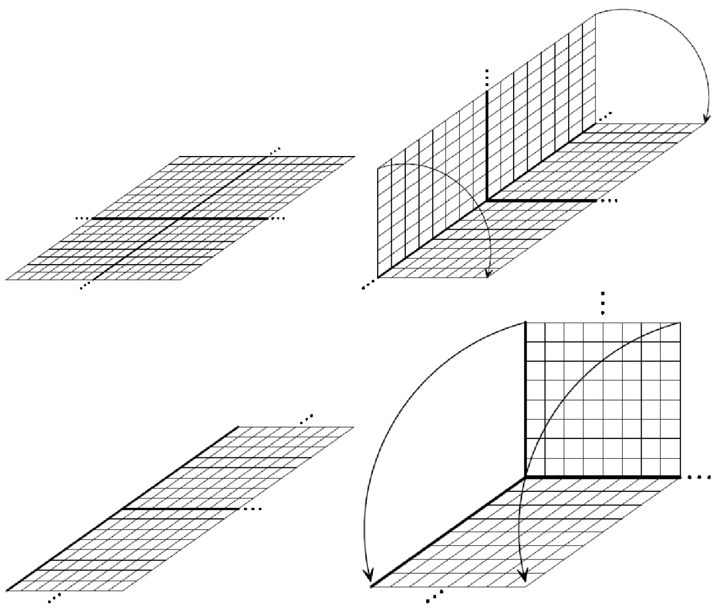

Figure 8: The mapping of example 5.4

length, as can be seen after a moments thought. It is not, however, open. The problem with this mapping, at least from our point of view, is that we can embed the square $[0,1] \times [0,1]$ to any manifold with dimension at least 2. Thus the composition of f and this embedding would allow us to map the Euclidean plane into any manifold with dimension at least 2. This means that the mapping f is way too flexible to be interesting.

Theorem 5.5. *The covering map $p_{\mathcal{M}}\colon \widehat{\mathcal{M}} \to \mathcal{M}$ of a length manifold $\mathcal{M}$ is 1-BLD.*

Proof. The covering map $p_{\mathcal{M}}$ is continuous by definition, and the path-length criterion is trivial as we have defined the path-length in $\widehat{\mathcal{M}}$ to be such that $p_{\mathcal{M}}$ preserves rectifiable paths and path lengths exactly. To see that it is a discrete mapping we take a point $x \in \mathcal{M}$, pick for it a covering neighbourhood and immediately we receive disjoint neighbourhoods for the points in $p_{\mathcal{M}}^{-1}\{x\}$ which shows that the set is discrete. To see that the mapping $p_{\mathcal{M}}$ is open, let us pick an open set $U \subset \widehat{\mathcal{M}}$. The mapping $p_{\mathcal{M}}$ is a local homeomorphism, so each point $x \in U$ has a neighbourhood V_x such that $V_x \subset U$ and $p_{\mathcal{M}}|_{V_x} \to p_{\mathcal{M}}[V_x]$ is a homeomorphism. Especially the sets $p_{\mathcal{M}}[V_x]$ are open, and thus the set $p_{\mathcal{M}}[U]$ is open as a union of the open sets $p_{\mathcal{M}}[V_x]$.

This means that the covering map is 1-BLD. □

5.2 Basic properties of discrete open continuous mappings

We shall need a few basic properties of BLD mappings later. In this section we prove some of them in their natural setting which is the situation of discrete open continuous mappings. Note that by our definition BLD mappings are always discrete, open and continuous. In this section and here alone we will be using results and terminology only used in appendix A.

In this section all discrete open mappings are assumed continuous.

Definition 5.6. The *branching set* B_f of a mapping f is defined to be the set

$$B_f = \{x \in \mathcal{M} \mid \text{The mapping } f \text{ is not a local homeomorphism at } x.\}$$

i.e. it is the set in which the mapping f fails to be a local homeomorphism.

Note that it follows directly from the definition that the branching set is closed, for if f is a local homeomorphism at $x \in \mathcal{M}$, then it is clearly a local homeomorphism is some neighbourhood of x. We state the following theorem proved by J. Väisälä in [Väi66] and its important corollary.

Theorem 5.7. *Suppose f is a discrete open map between two n-dimensional manifolds. Then the branching set of f has topological dimension at most $n-2$.*

Remark 5.8. A BLD mapping is always Lipschitz by lemma 5.3, and thus we have also $\dim f[B_f] = 0$ by basic properties of topological dimension.

Corollary 5.9. *The branching set of a open discrete map or its image under the same map cannot separate any two points on a manifold.*

Remark 5.10. Note that the only set with topological dimension -1 is the empty set, so a open discrete mapping between 1-dimensional manifolds is always a local homeomorphism.

We next need to show that paths can be lifted under open discrete maps. The idea behind the following results (lemma 5.11 and theorem 5.12) is from the proof of similar theorem found in [Ric93, Theorem 3.2., p.33]. In Rickman's book this theorem is proved for mappings between domains of $\mathbb{R}^n$ in a more general case which states that we can actually obtain a maximal amount of the pre-image of the path. We do not need such a strong result, so we shall prove the lemma separately.

The following proof is not only a restriction of Rickman's proof because in his theorem he assumes the mapping to be orientation-preserving between orientable manifolds. We do not need such assumptions. It suffices to have a open discrete mapping between manifolds as we need results concerning orientability only locally and manifolds are always locally orientable. The 'full version' of [Ric93, Theorem 3.2., p.33] could also be proven without orientable manifolds or maps with a similar proof as the one in this section.

In addition to all this, we prove a nice corollary of this theorem for BLD mappings with a complete domain. In this case we can have the *whole* path lifted and not just a maximal lift that we would have in the general case.

We first agree on some terminology. Let $f\colon X \to Y$ be a continuous mapping between two topological spaces and $\gamma\colon [0,1] \to Y$ a continuous path. We say that the path γ can be *lifted locally by* f, if the following holds. For any $y := \gamma(t_y) \in |\gamma|$, $x \in f^{-1}\{y\}$ there exists a path $\alpha\colon [a,b] \to X$ such that $\alpha(a) = x$, $b > a$ and $f \circ \alpha = \gamma|_{[t_y,t_b]}$ for some $t_b > t_y$.

The following lemma has been detached from the proof given by Rickman into a separate entity to clarify the idea behind the proof.

Lemma 5.11. *Let $\mathcal{M}$ and $\mathcal{N}$ be manifolds with $f\colon \mathcal{M} \to \mathcal{N}$ an open discrete map between them. If a path $\gamma\colon [0,1] \to f[\mathcal{M}]$ can be lifted locally by f, then it has a maximal lift via f starting from any point $f^{-1}\{\gamma(0)\}$.*

Furthermore, suppose $f\colon \mathcal{M} \to \mathcal{N}$ is a BLD mapping between length manifolds and that $\mathcal{M}$ is complete. Then any rectifiable path in $f[\mathcal{M}]$ has a total lift.

Proof. Assume $x_0 \in \mathcal{M}$, and that $\gamma\colon [0,1] \to f\,[\mathcal{M}]$, $\gamma(0) = f(x_0)$. We want to find a maximal lift with this root x_0.

Let us define Z to be the set of all paths $\gamma_\alpha\colon I_\alpha \to \mathcal{M}$ such that for all α we have that I_α is a subinterval of $[0,1]$ containing the element 0, $\gamma_\alpha(0) = x_0$ and $\gamma|_{I_\alpha} = f \circ \gamma_\alpha$. We define a partial order $\leq$ in the set Z by setting $\gamma_\alpha \leq \gamma_\beta$, if $I_\alpha \subset I_\beta$ and $\gamma_\beta|_{I_\alpha} = \gamma_\alpha$. The set Z is nonempty as we have a local lift by our assumption.

We see that if we take any ordered subset A of the set Z, we find an upper bound of this set by taking $I_0 = \bigcup\{I_\alpha \mid \gamma_\alpha \in A\}$ and defining $\gamma_0\colon I_0 \to \mathcal{M}$, $\gamma_0(t) = \gamma_\alpha(t)$, when $t \in I_\alpha$. This upper limit is well defined by the definitions of our partial order and the set A. But as an arbitrary ordered subset of Z has an upper bound, we have by Zorn's lemma that the partially ordered set $(Z, \leq)$ has a maximal element $\hat{\gamma}\colon I \to \mathcal{M}$.

If $I = [0,t_0]$ for some $0 < t_0 < 1$, we could take a neighbourhood of the point $\hat{\gamma}$ and lift the path γ further by our assumption. But this would contradict the maximality of the path $\hat{\gamma}$ in the set Z. Thus the only possibility is that $I = [0,1]$ or $I = [0,t[$ for some $0 < t < 1$. This is our maximal lift.

What we still need to show in the case of a BLD mapping between length manifolds and a complete domain we have $I = [0,1]$ if the path γ is rectifiable. By the previous argument the interval I can contain the rightmost endpoint if and only if $I = [0,1]$. If we would have that $I = [0,t_0[$ for some $0 < t_0 < 1$, we could pick a sequence (a_n) from I such that $\lim_{n\to\infty} a_n = t_0$. We note as γ is rectifiable, we can use the path length criterion of f to see that

$$\ell(\hat{\gamma}) \leq L\ell(f \circ \hat{\gamma}) = L\ell(\gamma) < \infty.$$

This means that $\hat{\gamma}$ is also rectifiable, so we have that

$$\sum_{k=1}^{\infty} d(\hat{\gamma}(a_k), \hat{\gamma}(a_{k+1})) \leq \sum_{k=1}^{\infty} \ell(\hat{\gamma}|_{[a_k,a_{k+1}]}) = \ell(\hat{\gamma}) < \infty, \tag{3}$$

and so the sequence $\hat{\gamma}(a_n)$ has to also be a Cauchy sequence because otherwise the leftmost sum in (3) would not converge. Especially the sequence $\hat{\gamma}(a_n)$

converges as $\mathcal{M}$ was assumed to be complete. So as $\lim_{n\to\infty} \hat{\gamma}(a_n) = a \in \mathcal{M}$, we can extend $\hat{\gamma}$ to $[0, t_0]$ by continuity of the mapping f. This would contradict the maximality of $\hat{\gamma}$, so the interval I can not be half-open and this proves the claim. $\square$

Note that in the last part of the proof we did not use the full power of the BLD criterion, just the fact that a BLD mapping maps rectifiable paths to rectifiable paths.

In the proof of the following theorem we rely heavily on the results proven in the appendix A. The proof of the following theorem is the only part (not counting the proof of 5.7) of this thesis that requires the cohomological results of the appendix. We also use terminology of orientation and topological index $i(x; f)$ that is defined only in the mentioned appendix.

Theorem 5.12. *Let $f\colon \mathcal{M} \to \mathcal{N}$ be a mapping between two manifolds. If $\beta\colon I \to f[N]$ is a path, then there exists a path $\tilde{\beta}\colon I \to M$ such that $f \circ \tilde{\beta} = \beta$ with any given starting point $x_0 \in f^{-1}\beta(0)$.*

The path $\tilde{\beta}$ is called the f-lift of β, *and is not usually unique.*

Proof. By lemma 5.11 we only need to show the existence of a local lift. We actually show a bit stronger claim: Every point $x \in \mathcal{M}$ has a neighbourhood A such that any path $\gamma\colon [0, 1] \to f[A]$ can be locally lifted under $f|_A$. We first pick a chart-neighbourhood V homeomorphic to $\mathbb{R}^n$ for the point $f(x_0)$. This neighbourhood will especially be orientable. Next we pick a chart-neighbourhood U for x_0 such that $U \subset f^{-1}[V]$. Now $f|_U\colon U \to V$ is a discrete open mapping between oriented manifolds. By choosing the orientations properly, we see can have $f|_U$ orientation-preserving, and by notions in appendix, we can have U so small (by taking it to be precompact) that the index is defined for all points in V. In the following we abuse notation by writing f instead of $f|_U$.

We shall prove the claim by induction over the index $i(x; f)$ of f in x.

Base step: If $i(x; f) = 1$, then the mapping f is an injection in a neighbourhood U of this point and as such a local homeomorphism. Thus we can define for any path $\beta\colon [0, 1] \to f[U]$ a lift $\tilde{\beta}\colon [0, 1] \to U$ by setting $\tilde{\beta} = f^{-1} \circ \beta$.

Inductive step: Assume the claim holds for any point x with $1 \le i(x; g) < r$, where $r \in \mathbb{N}$. Moreover, let us assume that $i(x_0; f) = r$ for $x_0 \in U$.

As $x \mapsto i(x; f)$ is upper semicontinuous by lemma A.14 we may assume that the neighbourhood U is so small that $i(x; f) \le r$ for all $x \in U$. Let us denote

$$F = \{z \in U \mid i(z; f) = r\}.$$

By lemma A.13 we know that the $f|_F$ is injective as U was chosen small enough. This means that for any path $\gamma\colon [0, 1] \to f[\mathcal{M}]$ we have for each $t \in \gamma^{-1}f[F]$ a unique $x_t \in U$ such that $f(x_t) = \gamma(t)$. So the part of the path γ that lies in $f[F]$ lifts uniquely. Now we only need to 'fill in the gaps'.

Again by the upper semicontinuity of the function $x \mapsto i(x; f)$, we know that the set F is closed in the set U. Because U was assumed precompact, $f|_U$ is a closed mapping, so $f[F]$ is closed in $f[U]$. This means that for any path $\gamma\colon [0, 1] \to f[\mathcal{M}]$ the set $\gamma^{-1}[f[U] \setminus f[F]]$ is a union of countably many disjoint intervals $]a_\lambda, b_\lambda[$, $\lambda \in \mathbb{N}$.

Let us fix the path γ for a moment. For each λ we pick $c_\lambda \in\,]a_\lambda, b_\lambda[$, and set $\gamma_\lambda = \gamma|_{[c_\lambda, b_\lambda[}$ and $\gamma'_\lambda = \gamma|_{]a_\lambda, c_\lambda]}$. The mapping $g = f|_{U\setminus F}$ meets the assumptions in the inductive step, so we know that we may lift locally any path $\alpha\colon [a,b] \to f[U] \setminus f[F]$ to a path in U. But this especially means that both γ_λ and γ'_λ can be lifted. (Strictly speaking our induction hypothesis only deals with paths from a closed interval, but we can approximate the half open interval with closed subintervals and lift the paths restricted to those to get the lift.)

Now we note that by continuity we must have that $\lim_{t\to b_\lambda} \gamma_\lambda(t) \in f[F]$ and $\lim_{t\to a_\lambda} \gamma'_\lambda(t) \in f[F]$, so we may now define our lift as a composition of lifts in intervals $]a_\lambda, c_\lambda[$ and on $\gamma^{-1} f[F]$. $\square$

With path-lifting we are able to prove the following useful theorem.

Theorem 5.13. *Let $\mathcal{M}$ and $\mathcal{N}$ be length-manifolds and $f\colon \mathcal{M} \to \mathcal{N}$ a BLD mapping. If $\mathcal{M}$ is complete, then f is surjective.*

Proof. Assume there exists a point $y \in \mathcal{N} \setminus \mathrm{Im}(f)$. Let us take a point $x \in \mathcal{M}$ and connect points $f(x)$ and y with a rectifiable path $\gamma\colon [0,1] \to \mathcal{N}$, $\gamma(0) = f(x)$, $\gamma(1) = y$. Define

$$t_0 := \sup\{t \in [0,1] \mid \gamma([0,t]) \subset \mathrm{Im}(f)\},$$

and note that we must have $\gamma(t_0) \notin \mathrm{Im}(f)$ as the mapping f was supposed open, so especially $\mathrm{Im}(f)$ has to be open.

Let us take a sequence (a_n) from $[0, t_0[$ that converges to t_0. By previous theorem, the path $\gamma_{[0,t_0[}$ lifts to a path $\hat{\gamma}\colon [0, t_0[\to \mathcal{M}$. Because the path $\hat{\gamma}$ is rectifiable, the sequence $(b_n), b_n = \hat{\gamma}(a_n)$ has to be a Cauchy sequence. Because we assumed the domain to be complete, the sequence must converge to a point $b \in \mathcal{M}$ [5]. But this would enable us to continue the lift of the path γ to the interval $[0, t_0]$, which would by continuity force us to have $f(b) = \gamma(t_0)$, which is a contradiction. $\square$

5.3 Growth-rate properties of BLD-mappings

We are now prepared with enough auxiliary results to start formulating and proving our main theorems concerning the non-existence of BLD mappings between manifolds in certain situations. Most of the results will be acquired by noting that BLD mappings are always Lipschitz quotient maps and using theorem 3.47 that restricts the existence of coarse Lipschitz quotient mappings by comparing growth rates.

The proof of the following theorem bears some resemblance to the proof of theorem 4.18. Connection between BLD and Lipschitz quotient mappings (with slightly different definitions) has been studied in [HR02].

Theorem 5.14. *Suppose $\mathcal{M}$ and $\mathcal{N}$ are two length manifolds and let $\mathcal{M}$ be complete. Then a BLD mapping $f\colon \mathcal{M} \to \mathcal{N}$ is Lipschitz quotient.*

Proof. Let $x_0 \in \mathcal{M}$. We will show that for any $r > 0$

$$\mathrm{B}\left(f(x_0), \frac{r}{L}\right) \subset f\left[\mathrm{B}\left(x_0, r\right)\right] \subset \mathrm{B}\left(f(x_0), Lr\right),$$

where L is the BLD-constant of f. Remember that f is surjective by theorem 5.13.

[5] This was done explicitly in the proof of lemma 5.11.

The second inclusion is easy, as we have already seen that an L-BLD mapping is always an L-Lipschitz mapping.

For the first inequality, let us assume the contrary; there exists a point

$$y_0 \in \mathrm{B}\left(f(x_0), \frac{r}{L}\right) \setminus f\left[\mathrm{B}\left(x_0, r\right)\right].$$

Because $y_0 \in \mathrm{B}\left(f(x_0), \frac{r}{L}\right)$, there exists a path γ in $\mathcal{N}$ connecting the points $f(x_0)$ and y_0 with $\ell(\gamma) < \frac{r}{L}$. Because the mapping f is surjective, the path γ can be lifted by lemma 5.11. Thus we have a lifted path $\tilde{\gamma}\colon [0,1] \to \mathcal{M}$ such that $f \circ \tilde{\gamma} = \gamma$.

Now we see that as $\gamma(0)$ lies in the interior and $\gamma(1)$ in the exterior of $f\left[\mathrm{B}\left(x_0, \frac{r}{L}\right)\right]$, we must have that $\tilde{\gamma}(0)$ lies in the interior and $\tilde{\gamma}(1)$ lies in the exterior of $\mathrm{B}\left(x_0, L\right)$. Thus by continuity of $\tilde{\gamma}$ there exists a point $y \in |\tilde{\gamma}| \cap \partial\,\mathrm{B}\left(x_0, L\right)$. Let $t \in [0,1]$ be a point such that $\tilde{\gamma}(t) = y$, and let us denote $\tilde{\gamma}_1 = \tilde{\gamma}_{[0,t]}$.

But now as $\tilde{\gamma}(t) \in \partial\,\mathrm{B}\left(x_0, L\right)$, we must have that $\ell(\tilde{\gamma}_1) \geq r$. On the other hand by the BLD criterion for f we have that

$$\ell(\tilde{\gamma}_1) \leq \ell(\tilde{\gamma}) \leq L\ell(f \circ \gamma) + 0 = L\ell(\gamma) < L\frac{r}{L} = r,$$

so $r \leq \ell(\gamma) < r$. This is a contradiction, so the original claim holds true. $\square$

Theorem 5.15. *Let $\mathcal{M}$ and $\mathcal{N}$ be two path-metric manifolds, $\left(\widehat{\mathcal{M}}, p_{\mathcal{M}}\right)$, $\left(\widehat{\mathcal{N}}, p_{\mathcal{N}}\right)$ their respective covers and $f\colon \mathcal{M} \to \mathcal{N}$ an L-BLD mapping. If a lift $f\colon \mathcal{M} \to \widehat{\mathcal{N}}$ exists, it is also an L-BLD mapping.*

If the double lift $\widetilde{\widetilde{f}}\colon \widehat{\mathcal{M}} \to \widehat{\mathcal{N}}$ exists it is also an L-BLD mapping. Especially BLD mappings always lift to BLD mappings between universal covers.

Proof. As the double lift f is by definition a lift of the composition of a f and a covering map, it will suffice to show that a lift of a L-BLD mapping is L-BLD and that the composition $f \circ p_{\mathcal{M}}\colon \widehat{\mathcal{M}} \to \mathcal{N}$ is an L-BLD map.

Let $\gamma\colon [0,1] \to \mathcal{M}$ be a rectifiable path and assume there exists a lift of f to to covering space $(\widehat{\mathcal{M}}, p_{\mathcal{M}})$ of $\mathcal{M}$. We note that by the definition of path length in covers we have

$$\ell_{\widehat{\mathcal{M}}}(\widetilde{f} \circ \gamma) = \ell_{\mathcal{M}}(p_{\mathcal{M}} \circ \widetilde{f} \circ \gamma) = \ell_{\mathcal{M}}(f \circ \gamma),$$

so the path-length criterion is satisfied for $\widetilde{f}$ exactly when it is satisfied for f. We also note that as $p_{\mathcal{M}}$ is a local homeomorphism, we can locally write $\widetilde{f} = f \circ (p_{\mathcal{M}}|_U)^{-1}$ for a suitable U. This means that $\widetilde{f}$ is continuous and open exactly when f is, as these are both local properties. Finally we note that if we pick any point $x \in \widehat{\mathcal{M}}$, we have that $\widetilde{f}^{-1}\{x\} \subset f^{-1}\{p_{\mathcal{M}}(x)\}$ and as a subset of a discrete set is always discrete we see that the mapping $\widetilde{f}$ is a discrete map.

Next we show that $f \circ p_{\mathcal{M}}$ is L-BLD. We have already showed in theorem 5.5 that the covering map $p_{\mathcal{M}}$ and is 1-BLD. The composition of two open continuous functions is open and continuous, so we only need to show discreteness and the path-criterion for the mapping $f \circ p_{\mathcal{M}}$. Let $\gamma\colon [0,1] \to \widetilde{\mathcal{M}}$ be a rectifiable path. As the mapping $p_{\mathcal{M}}$ is 1-BLD, the path $p_{\mathcal{M}} \circ \gamma$ is still rectifiable as is $(f \circ p_{\mathcal{M}}) \circ \gamma$. Moreover

$$\frac{1}{L}\ell\left(\gamma\right) \leq \frac{1}{L}\ell\left(p_{\mathcal{M}} \circ \gamma\right) \leq \ell\left(f \circ p_{\mathcal{M}} \circ \gamma\right) \leq L\ell\left(p_{\mathcal{M}} \circ \gamma\right) \leq L\ell\left(\gamma\right).$$

Thus the function $f \circ p_{\mathcal{M}}$ satisfies the path-criterion.

Let now $y_0 \in \mathcal{N}$. The set $f^{-1}\{y\}$ is discrete as f is. We wish to show that the set $p_{\mathcal{M}}^{-1}\left[f^{-1}\{y\}\right]$ is also discrete, but this actually follows immediately from the fact that the mapping $p_{\mathcal{M}}$ is a local homeomorphism.

□

Theorem 5.16. *Let $\mathcal{M}$ be a length manifold and $f\colon \mathbb{R}^n \to \mathcal{M}$ a BLD mapping. Then $\mathcal{M}$ is weakly doubling.*

Proof. We know theorem 5.14 that f is a Lipschitz quotient. This means that there exists constants C and r_0 such that

$$\mathrm{B}\left(f(x), C^{-1}r\right) \subset f\left[\mathrm{B}\left(x, r\right)\right] \subset \mathrm{B}\left(f(x), Cr\right)$$

for all $r \geq r_0$, $x \in \mathbb{R}^n$.

Suppose $\mathrm{B}\left(y, R\right) \subset \mathcal{M}$, and that there is a collection $\mathcal{B}$ of disjoint balls with radius $r < R$ within the ball $\mathrm{B}\left(y, R\right)$. Now there exists a ball $\mathrm{B}\left(x, CR\right) \subset \mathbb{R}^n$ whose image under f covers $\mathrm{B}\left(y, R\right)$. For each ball B in $\mathcal{B}$ we find a smaller ball with radius $C^{-1}r$ from $\mathbb{R}^n$ that is mapped within B. These small balls in $\mathbb{R}^n$ are all disjoint by their definition, so we have an upper bound to the amount of elements in $\mathcal{B}$ inherited by the weakly doubling constant of $\mathbb{R}^n$. □

5.4 Main results

Next we look some ways to apply our result. The following give restrictions to the existence of BLD mappings between certain manifolds by noting that BLD mappings induce Lipschitz quotient mappings between their respective universal covers when the domain is complete. Moreover these Lipschitz quotient mappings can be composed with coarse quasi-isometries to produce coarse Lipschitz quotient maps between any metric spaces that are coarsely quasi-isometric to the respective universal covers (see figure 9). An important example of such metric spaces is given by theorem 4.35 that tells that under good enough assumptions the product $\mathcal{M} \times \Pi_1(\mathcal{M})$ is coarsely quasi-isometric to $\widetilde{\mathcal{M}}$. From these we get nice corollaries in the cases when one of the factors in the product disappears in a coarse sense.

We state our main results with respect to the growth classes of length manifolds. In next section we shall take advantage from the results that give us information on whether or not there exists minimal elements in the growth class of a metric space. In this section we shall by $\mathrm{Ord}_*(X) \geq \mathrm{Ord}_*(Y)$ mean that for any net S in X we find a net P in Y such that $\mathrm{Ord}(S) \geq \mathrm{Ord}(P)$.

The following is our generalized Varopoulos type result:

Theorem 5.17. *Let $\mathcal{M}$ and $\mathcal{N}$ be length manifolds with $\mathcal{M}$ complete. Suppose there exists a BLD mapping $f\colon \mathcal{M} \to \mathcal{N}$. Then*

$$\mathrm{Ord}_*(\mathcal{M}) \geq \mathrm{Ord}_*(\mathcal{N}) \text{ and} \tag{4}$$

$$\mathrm{Ord}_*(\widetilde{\mathcal{M}}) \geq \mathrm{Ord}_*(\widetilde{\mathcal{N}}). \tag{5}$$

Moreover, whenever the mapping f lifts between covers $\widehat{\mathcal{M}}$ and $\widehat{\mathcal{N}}$ of $\mathcal{M}$ and $\mathcal{N}$, respectively, we must have $\mathrm{Ord}_*(\widehat{\mathcal{M}}) \geq \mathrm{Ord}_* \widehat{\mathcal{N}}$.

$$
\begin{tikzcd}
\mathcal{M} \times \Pi_1(\mathcal{M}) \arrow[r, "\hat{f}", "\text{coarse Lipschitz quotient}"'] \arrow[d, squiggly, "\varphi_{\mathcal{M}}"] & \mathcal{N} \times \Pi_1(\mathcal{N}) \\
\widetilde{\mathcal{M}} \arrow[r, "\tilde{\tilde{f}}", "\text{Lipschitz Quotient}"'] \arrow[d, "p_{\mathcal{M}}"] & \widetilde{\mathcal{N}} \arrow[u, squiggly, "\varphi_{\mathcal{N}}"'] \arrow[d, "p_{\mathcal{N}}"] \\
\mathcal{M} \arrow[r, "f", "\text{BLD}"'] & \mathcal{N}
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "\hat{f}", "\text{coarse Lipschitz quotient}"'] \arrow[d, squiggly, "\xi_{\mathcal{M}}"] & Y \\
\widetilde{\mathcal{M}} \arrow[r, "\tilde{\tilde{f}}", "\text{Lipschitz Quotient}"'] \arrow[d, "p_{\mathcal{M}}"] & \widetilde{\mathcal{N}} \arrow[u, squiggly, "\xi_{\mathcal{N}}"'] \arrow[d, "p_{\mathcal{N}}"] \\
\mathcal{M} \arrow[r, "f", "\text{BLD}"'] & \mathcal{N}
\end{tikzcd}
$$

Figure 9: The fundamental idea behind our main results.

Proof. A BLD mapping $f : \mathcal{M} \to \mathcal{N}$ from a complete manifold is always Lipschitz quotient by theorem 5.14. By theorem 3.47 this means that $\mathrm{Ord}_*(\mathcal{M}) \geq \mathrm{Ord}_*(\mathcal{N})$.

By theorem 5.15 any lift of a BLD mapping $f : \mathcal{M} \to \mathcal{N}$ is still a BLD mapping. This implies the rest of the claim. □

Theorem 5.18. *Let $f : \mathcal{M} \to \mathcal{N}$ be a BLD mapping between length manifolds with $\mathcal{M}$ complete. If X and Y are metric spaces that are coarsely quasi-isometric to $\mathcal{M}$ and $\mathcal{N}$, respectively, we must have* $\mathrm{Ord}_*(X) \geq \mathrm{Ord}_*(Y)$.

Proof. This follows from the fact that the mapping $\hat{f} : X \to Y$ acquired by conjugating the mapping f with the given coarse quasi-isometries is coarse Lipschitz quotient by theorem 3.45, and by theorem 3.47 we must have $\mathrm{Ord}_*(X) \geq \mathrm{Ord}_*(Y)$. □

Corollary 5.19. *Let $\mathcal{M}$ and $\mathcal{N}$ be length manifolds such that $\mathcal{M}$ is complete. Suppose there exists a BLD mapping $f : \mathcal{M} \to \mathcal{N}$.*

a) If $\mathcal{N}$ is geometrically homogeneous, then $\Pi_1(\mathcal{N})$ is finitely generated, and we must have

$$\mathrm{Ord}_*(\widetilde{\mathcal{M}}) \geq \mathrm{Ord}_*(\widetilde{\mathcal{N}}) = \mathrm{Ord}_*(\mathcal{N} \times \Pi_1(\mathcal{N}))$$

b) If $\mathcal{M}$ is geometrically homogeneous, then $\Pi_1(\mathcal{M})$ is finitely generated, and we must have

$$\mathrm{Ord}_*(\widetilde{\mathcal{N}}) \leq \mathrm{Ord}_*(\widetilde{\mathcal{M}}) = \mathrm{Ord}_*(\mathcal{M} \times \Pi_1(\mathcal{M}))$$

c) If $\mathcal{M}$ and $\mathcal{N}$ are both geometrically homogeneous, then $\Pi_1(\mathcal{M})$ and $\Pi_1(\mathcal{N})$ are finitely generated, and we must have

$$\mathrm{Ord}_*(\mathcal{M} \times \Pi_1(\mathcal{M})) \geq \mathrm{Ord}_*(\mathcal{N} \times \Pi_1(\mathcal{N}))$$

Furthermore, suppose a manifold X is weakly doubling and has a finitely generated fundamental group. Then the product $X \times \Pi_1(X)$ is also weakly doubling and we can calculate the growth of a product by product of growths:

$$\mathrm{Ord}(X \times \Pi_1(X)) = \mathrm{Ord}(X) \cdot \mathrm{Ord}(\Pi_1(X)).$$

5.5 Changing between metrics

We have shown thus far all our results in the case where the metric used is a path-metric. We wish now to check how much the metric can be changed without our theorems coming void. This is a very natural question, as for different metrics our non-existence -results change dramatically. There is, for example, no BLD mapping from $\mathbb{R}^2$ to a genus 2 surface as we soon show. We can, however map the open unit ball onto a genus 2 surface with a BLD map even though the unit disk and the plane are homeomorphic and either one can be given a metric with which they are even isometric.

We shall next formulate and prove a result that states that our results still hold if we change our path-metric into another metric that is bi-Lipschitz-equivalent with the original metric.

Theorem 5.20. *Let $f(\mathcal{M}, d) \to (\mathcal{N}, d')$ be a BLD mapping between two length manifolds. Suppose we change metrics d and d' into bi-Lipschitz-equivalent metrics $\hat{d}$ and $\hat{d}'$, respectively. Then the mapping $f\colon (\mathcal{M}, \hat{d}) \to (\mathcal{N}, \hat{d}')$ is still a BLD mapping.*

Proof. Continuity, discreteness and openness are topological properties and are not affected by the change into an topologically equivalent metric. Checking the path-length criterion requires a bit more work.

Remember from definition 4.2 that we can equip any path-connected metric space with some sort of path-length structure. If we use this method to a length manifold we receive the same length structure we already had. So if d and $\hat{d}$ are L-bi-Lipschitz-equivalent, then for all partitions $(a_1, \ldots, a_n)$ of the unit interval we have that

$$\frac{1}{L}\sum_{i=1}^{n-1} \hat{d}\left(\gamma(a_i), \gamma(a_{i+1})\right) \leq \sum_{i=1}^{n-1} d\left(\gamma(a_i), \gamma(a_{i+1})\right) \leq L\sum_{i=1}^{n-1} \hat{d}\left(\gamma(a_i), \gamma(a_{i+1})\right),$$

so especially

$$\frac{1}{L}\ell_{\hat{d}}(\cdot) \leq \ell_d(\cdot) \leq L\ell_{\hat{d}}(\cdot).$$

From this we also see that the set of rectifiable paths does not change under the change of the metric.

Let us denote by L the BLD constant of f with respect to the metrics d and d', by K_1 the constant that represents the bi-Lipschitz change of metric d to $\hat{d}$, and by K_2 the constant associated with the change $d' \mapsto \hat{d}'$. Now we see that by using this length-preservance of bi-Lipschitz metric change, the BLD property of f with respect to the metrics d and d', and then again the length-preservance of bi-Lipschitz metric change we have that

$$\ell_{\hat{d}'}(f \circ \gamma) \leq K_1 \ell_{d'}(f \circ \gamma) \leq LK_1 \ell_d(\gamma) \leq K_2 L K_1 \ell_{\hat{d}}(\gamma)$$

and

$$\ell_{\hat{d}'}(f \circ \gamma) \geq \frac{1}{K_1}\ell_{d'}(f \circ \gamma) \geq \frac{1}{LK_1}\ell_d(\gamma) \geq \frac{1}{K_2 L K_1}\ell_{\hat{d}}(\gamma).$$

Thus we have that with respect to the new metrics $\hat{d}$ and $\hat{d}'$ the mapping f is a $(K_2 L K_1)$-BLD mapping.

□

By imitating the proof of the previous theorem, one easily sees that also the concepts of growth rate, weakly doublingness, null-homotopy radius, diameter bound and homogeneous geometry are practically invariant under bi-Lipschitz changes of metrics.

5.6 Concrete examples

We now turn to some concrete applications of our theorems. In this section we give fewer details and use less rigor as we wish to concentrate on giving ideas about what kind of things our theorems can be used to prove.

We showed in theorem 3.25 that $\mathbb{R}^n$ is weakly doubling, so it has a unique growth rate. It is easy to calculate that the growth of the 1-net $\mathbb{Z}^n$ of $\mathbb{R}^n$ has growth rate $\mathcal{O}(x^n)$. Thus $\mathrm{Ord}(\mathbb{R}^n) = \mathcal{O}(x^n)$. Moreover, if we have a BLD mapping $f\colon \mathbb{R}^n \to \mathcal{M}$, then $\mathcal{M}$ is weakly doubling and its growth rate is easy to calculate. Also remember that all finitely generated groups are weakly doubling 1-nets, so their growth rates are unique and easy to calculate.

Example 5.21. Suppose $\mathcal{N}$ is a compact n-dimensional length manifold and $\mathrm{Ord}(\Pi_1(\mathcal{N})) \geq \mathcal{O}(x^d)$ where $d > n$. Then there exists no BLD mapping from $\mathbb{R}^n$ to $\mathcal{N}$.

Example 5.22. Suppose $\mathcal{N}$ is a compact n-dimensional length manifold and $\mathrm{Ord}(\Pi_1(\mathcal{N})) \geq \mathcal{O}(x^d)$ where $d > n$. Then there exists No BLD mapping from $\mathbb{R}^{n+k}$ to $\mathbb{R}^k \times \mathcal{N}$.

Example 5.23. There cannot exist a BLD mapping from a compact length manifold to an unbounded length manifold.

Let $\mathcal{M}$ and $\mathcal{N}$ be two n-dimensional manifolds. The *surgery* of these two is denoted by $\mathcal{M}\#\mathcal{N}$ and denoted as follows. Take two sets $U_{\mathcal{M}} \subsetneq \mathcal{M}$ and $U_{\mathcal{N}} \subsetneq \mathcal{N}$ homeomorphic to the Euclidean n-dimensional unit ball. (Such sets exist as $\mathcal{M}$ and $\mathcal{N}$ are manifolds.) We remove these sets from the respective manifolds and take a cylinder $[0,1] \times S^1$. Now we 'glue' the ends $\{0\} \times S^1$ to $\partial U_{\mathcal{M}}$ and $\{1\} \times S^1$ to $\partial U_{\mathcal{N}}$. The result is a join of the two original manifolds.

If $\mathcal{M}$ and $\mathcal{N}$ are weakly doubling, then so is $\mathcal{M}\#\mathcal{N}$, and if $\mathcal{O}(f) \geq \mathrm{Ord}(\mathcal{M})$ and $\mathcal{O}(f) \geq \mathrm{Ord}(\mathcal{N})$, then $\mathcal{O}(f) \geq \mathrm{Ord}(\mathcal{M}\#\mathcal{N})$.

The following could be called a weak Jormakka type result.

Example 5.24. Let $\mathcal{M}$ and $\mathcal{N}$ be two length spaces. If

$$\Pi_1(\mathcal{M}) \neq \{0\} \neq \Pi_1(\mathcal{N}) \neq \mathbb{Z}_2,$$

Then $\mathcal{M}\#\mathcal{N}$ contains a free group spanned by two elements and there is no BLD mapping $\mathbb{R}^3 \to \mathcal{M}\#\mathcal{N}$.

We can have strict inequalities in our growth rate changes as the following inequalities show.

Example 5.25. We can map both the torus and the plane to S^2.

Example 5.26. There exists a BLD mapping from an orientable genus two surface to torus.

Example 5.27. The completeness assumption is essential, as we note that we can map the bounded simply connected manifold $]0,1[\times]0,1[$ subjectively to the open annulus $\mathrm{B}\left(\overline{0},2\right)\setminus\overline{\mathrm{B}}(\overline{0},1)$ via stretching and the complex exponential map.

This shows that the requirement of a complete domain is indispensable as the universal cover of the annulus has linear growth rate, even though the domain is bounded. (In general, BLD mappings with non-complete domains need not be surjective and our arguments concerning growth rates fail.)

Example 5.28. Let $\mathcal{M}$ be a length manifold, and $\widehat{\mathcal{M}}$ its cover. Then $\mathrm{Ord}_*(\mathcal{M}) \leq \mathrm{Ord}_*(\widehat{\mathcal{M}})$ as the covering map is a BLD mapping.

A dual statement that follows from the fact that fundamental groups of covering spaces of a manifold $\mathcal{M}$ are subgroups of $\Pi_1(\mathcal{M})$ is the following. Let $\mathcal{M}$ be a manifold with a finitely generated fundamental group and $\widehat{\mathcal{M}}$ its cover such that $\Pi(\widehat{\mathcal{M}})$ is finitely generated. Then $\mathrm{Ord}_*(\Pi_1(\mathcal{M})) \geq \mathrm{Ord}_*(\Pi_1(\hat{\mathcal{M}}))$.

From previous statements we know the following. Let $\mathcal{M}$ be a geometrically homogeneous manifold and $\widehat{\mathcal{M}}$ its cover. Then

$$\mathrm{Ord}_*(\mathcal{M} \times \Pi_1(\mathcal{M})) = \mathrm{Ord}_*(\widehat{\mathcal{M}} \times \Pi_1(\widehat{\mathcal{M}})).$$

If the manifold $\mathcal{M}$ also is weakly doubling we have

$$\mathrm{Ord}(\mathcal{M})\,\mathrm{Ord}(\Pi_1(\mathcal{M})) = \mathrm{Ord}(\widehat{\mathcal{M}})\,\mathrm{Ord}(\Pi_1(\widehat{\mathcal{M}})).$$

A Cohomology and topological index

In this appendix we go first through a short exposition of singular cohomology with compact supports to be able to give some basic results of index theory. The definition of singular cohomology with compact supports is standard, and can be found in any basic book dealing with cohomology theory. We also give references to sources that tell us that the choice of a specific cohomology theory is irrelevant for a manifold.

A.1 Singular cohomology of topological spaces with compact supports

We need a lemma (lemma A.13) that gives us local injectivity of a the restriction of an open discrete map to its branching set in section 5.2 when proving Rickman's theorem (theorem 5.12) concerning the lifting of paths under open discrete mappings. To prove this we need to prove some index theorems, which in turn require (local) topological orientation. To define topological orientation some sort of homology- or cohomology theory must be defined. We will in this thesis go with cohomology theory, for the index theorem we need is natural to prove in this setting. By for example [Spa66, Corollary 7, p. 341], [Spa66, Corollary 8, p. 334] or [War83, Theorem 5.23, p. 181] we know that for manifolds the choice of a specific cohomology theory does not make any difference. We shall use the so called singular cohomology with compact supports because the definition is dual to the definition of a singular homology theory which happens to be a quite natural and mechanically of not too complex structure. We give next the definition and the most important basic properties of singular cohomology with compact supports (and on the way we will also come to define singular homology and singular cohomology without compact supports). The form of the definitions is motivated by [Hat02].

We first define an abstract concept of a homology of a chain complex. Then we construct suitable chain complexes to obtain the required structures.

Definition A.1. A *chain complex* $\mathcal{C}$ is a sequence of abelian groups $C_n, n \in \mathbb{Z}$

$$\cdots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

together with *boundary* mappings $\partial_n \colon C_n \to C_{n-1}$ such that $\operatorname{Im} \partial_{n+1} \subset \operatorname{Ker} \partial_n$ for all $n \in \mathbb{Z}$.

The *homology groups* of this complex are defined as the quotient groups

$$H_n(\mathcal{C}) := {\operatorname{Ker} \partial_n}\Big/{\operatorname{Im} \partial_{n-1}}\,.$$

Definition A.2. Let $\mathcal{C}$ be a chain complex, and let G be an abelian group. The *cochain complex* of $\mathcal{C}$ with coefficients in G is the chain complex

$$\cdots \xleftarrow[\delta_{n+1}]{} C^{n+1} \xleftarrow[\delta_n]{} C^n \xleftarrow[\delta_{n-1}]{} C^{n-1} \xleftarrow[\delta_{n+2}]{} \cdots\,,$$

where

$$C^n = \operatorname{Hom}(C_n, G) := \{g \colon C_n \to G \mid g \text{ is a homomorphism }\}\,,$$

and the mappings δ_n are defined by $\delta_n(f) = f \circ \partial_{n+1}$.

$$\begin{tikzcd}
C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[dr, dotted, "\delta_n f := f \circ \partial_{n+1}"'] & C_n \arrow[d, "f"] \\
 & G
\end{tikzcd}$$

The structure of a cochain complex can be seen in picture 10.

The homology of the cochain complex with coefficients in the group G is called the *cohomology* of the original chain complex and denoted $H^*(\mathcal{C}; G)$.

$$\begin{tikzcd}
 & C^{n+1} & C^{n} & C^{n-1} & \\
\cdots \arrow[r, "\partial_{n+2}"] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d] & C_n \arrow[r, "\partial_n"] \arrow[d] & C_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d] & \cdots \\
 & G & G \arrow[l, dotted, "\delta_n"] & G \arrow[l, dotted, "\delta_{n-1}"] &
\end{tikzcd}$$

Figure 10: Structure of the cochain complex.

We next define a chain complex from which we can extract the singular homology theory and from its cochain complex the singular cohomology theory. To construct all this we need first some basic concepts.

A *standard n-simplex* is defined to be

$$\Delta_n = \big\{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1, t_i \geq 0 \text{ for all } i\big\}.$$

An *n-simplex* is a pair (S, ψ) with S a topological space and $\psi \colon \Delta_n \to S$ a homeomorphism. The *k-face* of a n-simplex S, $k \leq n+1$, with $\psi \colon \Delta_n \to S$ as a homeomorphism is the set

$$\psi\big[\big\{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum_i t_i = 1, t_i \geq 0 \text{ for all } i, t_k = 0\big\}\big]$$

and it is a $(n-1)$-simplex in a natural way. The k-face of a standard n-simplex we denote by $[v_0, \ldots, \hat{v_k}, \ldots, v_n]$. The *vertexes* of an n-simplex S, with $\psi \colon \Delta_n \to X$ as a homeomorphism, are the points $v_j = \psi\big((0, \ldots, \underbrace{1}_{i\text{:th}}, \ldots, 0)\big)$. We say that an n-simplex (S, ψ) is an n-simplex *of* a space X if $S \subset X$.

We call any ordering of the vertexes of an n-simplex an orientation. This ordering is inherited to the faces by a restriction of the orientation. For example the orientation of the k-face of an n-simplex with the orientation

$$(v_{i_1}, \ldots, v_{i_k}, \ldots, v_{i_n})$$

is just

$$(v_{i_1}, \ldots, v_{i_{k-1}}, v_{i_{k+1}}, \ldots, v_{i_n}).$$

A *singular n-simplex* of a topological space X is any continuous mapping $f\colon \Delta_n \to X$. We denote by $C_n^S(X)$ the free $\mathbb{Z}$-module generated by all singular n-simplexes of a topological space X. We make the identification

$$(v_1, \dots, v_i, \dots, v_j, \dots, v_n) = -(v_1, \dots, v_j, \dots, v_i, \dots, v_n).$$

The module $C_n^S(X)$ forms a chain complex when we define the boundary operators ∂_n by setting

$$\partial_n(f) = \sum_{k=0}^{n} (-1)^k f|_{[v_0, \dots, \hat{v_k}, \dots, v_n]} \tag{6}$$

for each singular n-simplex f.

Definition A.3. The *singular chain complex* of a topological space X, denoted $\mathcal{C}^S(X)$, is the chain complex constructed from the groups $C_n^S(X)$ and the mapping defined by (6). Its homology is called the *singular homology* of the space X and it is denoted $H_*(X)$. The elements of the groups in the singular chain complex are called *chains*.

Definition A.4. The *singular cohomology* of a topological space X with coefficients in an abelian group G is the cohomology of the singular chain complex $\mathcal{C}^S(X)$, and it is denoted $H^*(X; G)$.

The groups forming the cochain complex are denoted by $C_S^n(X; G)$ and their elements are called cochains.

Definition A.5. Let $\tilde{C}_S^n(X; G) \leq C_S^n(X; G)$ be the subgroup that consists of those mappings $\varphi\colon C_i(X) \to G$ for which there exists a compact set $K_\varphi \subset X$ such that φ is zero on all chains that lie in $X \setminus K_\varphi$.

From these subgroups we acquire a 'subchain complex' of the singular cochain complex and the homology of this chain complex is called the *singular cohomology with compact supports* of the space X and it is denoted by $H_c^*(X; G)$.

Definition A.6. Let $f\colon X \to Y$ be a continuous proper mapping between topological spaces. The *pullback of f* with respect to the singular cohomology with compact supports is the mapping $f^*\colon H_c^n(Y) \to H_c^n(X)$ defined as follows. If $\xi \in [\omega] \in H_c^n(Y)$, and $g \in C_n(Y)$, then $(f^*\xi)(g) = \xi(f \circ g)$. This induces a well-defined mapping between the cohomology groups with compact supports.

If we have an inclusion $\iota\colon A \hookrightarrow U$, we define also the push-forward

$$\iota\colon H_c^n(A) \to H_c^n(U)$$

by setting

$$(\iota_*\xi)(g) = \begin{cases} \xi(g), & \text{if } \mathrm{Im}(g) \subset A, \\ e_G, & \text{otherwise.} \end{cases}$$

for all singular n-simplexes. (This uniquely determines $\iota_*\xi$ and thus the whole of ι_*. This also induces a well-defined mapping between the cohomology groups with compact support.)

A.2 Topological index

In this section $H_c^n(\mathcal{M})$ will mean at all times the compactly supported singular cohomology of the space $\mathcal{M}$ with coefficients in $\mathbb{Z}$. The degree n in the notation $H_c^n(\mathcal{M})$ is assumed to equal the dimension of the manifold $\mathcal{M}$. All proper maps are assumed also continuous.

Our formulation for the definition of a topological index will follow the exposition given in [Väi66]. We will first define the concept of topological orientation on a manifold and then continue with the (local) topological index for *orientable* topological manifolds. The requirement causes no problems for us, as we need orientability only locally in and every manifold is locally orientable. (This can be seen for example by looking at the chart-neighbourhoods.)

We know for example by [Spa66] that for all manifolds we have either $H_c^n(\mathcal{M}) = \mathbb{Z}$ or $H_c^n(\mathcal{M}) = \mathbb{Z}_2$. these cases will be called the *orientable*- and the *non-orientable* case, respectively. From now on all manifolds are assumed orientable, and a spanning element $\mu_{\mathcal{M}}$ of $H_c^n(\mathcal{M})$ is called an *orientation* of the manifold. The following lemma allows us to orient all domains of our oriented manifold consistently. Proof can be found from [Spa66].

Lemma A.7. *Let $\mathcal{M}$ be an n-dimensional orientable topological manifold and $U \subset \mathcal{M}$ a domain. The mappings induced be the inclusion $\iota\colon U \hookrightarrow \mathcal{M}$,*

$$\iota_*\colon H_c^n(U) \to H_c^n(\mathcal{M}) \qquad \textit{and} \qquad \iota^*\colon H_c^n(\mathcal{M}) \to H_c^n(U),$$

are isomorphisms.

For any domain U of a manifold $\mathcal{M}$ we give an orientation via the mapping $\iota^*\colon H_c^n(\mathcal{M}) \to H_c^n(U)$ induced by the inclusion $\iota\colon U \hookrightarrow \mathcal{M}$ by setting $\mu_U := \iota^*(\mu_{\mathcal{M}})$.

If we have a proper map $f\colon (\mathcal{M}, \mu_{\mathcal{M}}) \to (\mathcal{N}, \mu_{\mathcal{N}})$ between oriented manifolds it induces a homomorphism between the compactly supported cohomologies $f^*\colon H_c^n(\mathcal{N}) \to H_c^n(\mathcal{M})$. As these are isomorphic to $\mathbb{Z}$ with $\mu_{\mathcal{N}}$ and $\mu_{\mathcal{M}}$ as spanning elements, we must have $f^*(\mu_{\mathcal{N}}) = k\mu_{\mathcal{M}}$ for some $k \in \mathbb{Z}$. We will say that f is *orientation preserving* if $k \geq 0$, otherwise we will say that it is *orientation reversing*.

We now turn to the index of a continuous open discrete mapping. Let $f\colon \mathcal{M} \to \mathcal{N}$ be a continuous open discrete mapping. Given a domain $U \subset \mathcal{M}$, a point $y \in \mathcal{N}$ is called (f, U)-admissible if there is a connected neighbourhood V of y such that f defines a proper mapping $f_p := f|_{U \cap f^{-1}[V]}\colon U \cap f^{-1}[V] \to V$. (We need proper mappings in order to get well-defined pull-back mappings between the compactly supported cohomologies.) For each (f, U)-admissible point y, we define the topological index $\mu(y, f, U)$ as follows. Take any neighbourhood V of y as above. From the inclusion $j\colon U \cap f^{-1}[V] \hookrightarrow V$ we get the homomorphism $j_*\colon H_c^n\left(U \cap f^{-1}V\right) \to H_c^n(U)$. Combining this with the (well-defined) pullback of f_p we have a mapping

$$j_* \circ f_p^*\colon H_c^n(V) \to H_c^n(U).$$

As the n:th cohomology groups with compact supports are $\mathbb{Z}$ for orientable n-dimensional manifolds, we note that we must have $\left(j_* \circ f_p^*\right)(\mu_V) = k\mu_U$ for some $k \in \mathbb{Z}$. This integer k is called the *topological index*. We denote it by $\mu(y, f, U)$. To see that the notation is sensible, we note that the topological index is independent of the choice of V. This follows as if we would have another neighbourhood V' of y that satisfies all the given conditions, then $V' \cap V$ would also satisfy these. As the inclusions

$$V' \cap V \hookrightarrow V \text{ and } \big(U \cap f^{-1}[V] \cap f^{-1}[V']\big) \hookrightarrow U \cap f^{-1}[V]$$

induce orientation preserving isomorphisms, we see that the indexes we receive from V and $V' \cap V$ equal. This shows that the index is independent of the choice of V.

To make the definition of topological index feasible, we want to see that it is not hard to find triples (U, f, y) such that the point y is (U, f)-admissible. The following lemma implies that for any $x \in \mathcal{M}$ we have a neighbourhood U such that $f(x)$ is (U, f)-admissible. The claim basically follows from [Väi66, Lemma 5.1, p.5]

Lemma A.8. *Let $f\colon \mathcal{M} \to \mathcal{N}$ be a continuous open discrete mapping. If $U \subset \mathcal{M}$ is a domain such that $\overline{U}$ is compact, then each point of the set $\mathcal{N} \setminus \partial f[U]$ is (U, f) admissible.*

We will have some dealing with the supports of singular cochains, and for that we agree on the following notation. To simplify typography, we write in shorthand $C^n_c(U) := \tilde{C}^n_S(U; \mathbb{Z})$. If we have $A \subset U$, then we denote

$$A^* = \{g \in C^n_c(U) \mid \mathrm{Im}(g) \subset A\}.$$

Also, if $\xi \in C^n_c(U)$, we set

$$\mathrm{supp}_*(\xi) = \bigcup \{\mathrm{Im}(g) \mid g \in \mathrm{supp}(\xi)\}$$

Note that

$$\mathrm{supp}_*(\xi) \subset \bigcup_i U_i \quad \text{if and only if} \quad \mathrm{supp}(\xi) \subset \bigcup_i U_i^*. \tag{7}$$

The following lemma is clear.

Lemma A.9. *Let $f\colon X \to G$ be a mapping from an arbitrary set to a group. If $\{U_i \mid i \in I\}$ is a collection of disjoint sets and $\mathrm{supp}(f) \subset \cup_{i \in I} U_i$, then*

$$f = \sum_{i \in I} f|_{U_i}.$$

In this context $f|_{U_i}$ is understood to be defined on the whole of X and be identically zero outside the set U_i.

Theorem A.10. *Let $U_1, \dots, U_k$ be disjoint open subsets of $\mathcal{M}$ and assume*

$$U \cap f^{-1}\{y\} \subset \bigcup_j U_j \subset U.$$

Then

$$\mu(y, f, U) = \sum_{i=1}^{k} \mu(y, f, U_i)$$

when y is admissible for all the pairs (f, U_i) and (f, U).

Remark A.11. What we really prove is expressed as follows.

$$" \mu(y,f,D)\mu_U = f^*\mu_V = \sum_i (f^*\mu_V)|_{U_i} = \sum_i (f|_{U_i})^*\mu_V = \sum_i \mu(y,f,U_i)\mu_U "$$

The problem is that most of this is ill-defined. The following proof mainly consists of bringing the correct formal manipulation to the idea behind this 'equation'.

Proof. Let $V \subset \mathcal{N}$ be the neighbourhood of y given in the definition of the topological index. By our assumptions we know that $f_p := f|_{U\cap f^{-1}[V]}$ is a proper mapping. From this it follows that we must have $\partial U \cap f^{-1}\{y\} = \emptyset$ because otherwise we could pick a small compact neighbourhood K of y, whose pre-image under f_p would the intersect both the open set U and its boundary. This cannot happen as then $f_p^{-1}[K]$ would not be compact even though f_p should be proper. We want to 'move' the restriction part of this mapping. This is why we need to the following: In the situation above we have that the mappings

$$\left(j_*(f_p^*\xi_V)\right)|_{U_i^*} \colon C_c^n(U_i) \to \mathbb{Z} \quad \text{and} \quad (j_*((f|_{U_i})^*\xi_V)) : C_c^n(U_i) \to \mathbb{Z} \tag{8}$$

equal. To see this let ω be a singular simplex, i.e. $\omega \colon \Delta_n \to U_i$. Now

$$\begin{aligned}
\left(j_*(f_p^*\xi_V)\right)|_{U_i^*}(\omega) &= j_*\left((f_p^*\xi_V)(\omega)\right) \\
&= (\xi_V)(f_p \circ \omega) \\
&= (\xi_V)(f|_{U_i} \circ \omega) \\
&= ((f|_{U_i})^*\xi_V)(\omega) \\
&= (j_*((f|_{U_i})^*\xi_V))(\omega),
\end{aligned}$$

which proves the claim.

Now finally combining the definition of topological index, equations (**??**) and (8), we get

$$\begin{aligned}
\mu(y,f,U)\xi_U &= \left(j_* \circ f_p^*\right)(\xi_V) \\
&= \sum_i \iota_*^i\left(\left((j_* \circ f_p^*)(\xi_V)\right)|_{U_i^*}\right) \\
&= \sum_i \iota_*^i\left((j_* \circ (f|_{U_i})^*)(\xi_V)\right) \\
&= \sum_i \iota_*^i\left(\mu(y,f|_{U_i},U_i)\xi_{U_i}\right) \\
&= \sum_i \mu(y,f|_{U_i},U_i)\iota_*^i\left(\xi_{U_i}\right) \\
&= \sum_i \mu(y,f|_{U_i},U_i)\xi_U \\
&= \sum_i \mu(y,f,U_i)\xi_U.
\end{aligned}$$

□

From this theorem it follows, that if U_1 and U_2 are two neighbourhoods of $x \in \mathcal{M}$ such that $\overline{U}_i \cap f^{-1}f(x) = \{x\}$, then $\mu(f(x), f, U_1) = \mu(f(x), f, U_1)$. Thus we may define that $i(x; f) = \mu(f(x), f, U)$ for any neighbourhood U of $x \in \mathcal{M}$ such that $\overline{U} \cap f^{-1}f(x) = \{x\}$.

Corollary A.12. *In the situation of the previous theorem, we have that*

$$\mu(y, f, U) = \sum_{w \in f^{-1}V \cap U} i(w, f).$$

Proof. This follows by choosing the sets U_i so small that $f^{-1}[f(x)] \cap U_i = \{x\}$ for all $x \in U_i$ for all i. □

The following lemma is the result we need in proving our path-lifting theorem 5.12.

Lemma A.13. *Suppose $f\colon \mathcal{M} \to \mathcal{N}$ is an orientation-preserving open discrete mapping between oriented manifolds. Then for any $r \geq 0$, every point*

$$y \in \{x \in \mathcal{M} \mid i(x; f) = r\} =: F$$

has a neighbourhood U such that $f|_{U \cap F}$ is injective.

Proof. As the mapping $x \mapsto i(x; f)$ is upper semicontinuous, we can pick for any $x_0 \in F$ a neighbourhood U such that the index is well defined and $i(x; f) \leq i(x_0; f)$ for all $x \in U$.

Let $x \in F \cap U$. By assumptions on U and orientability of f we know that for all $w \in f^{-1}\{f(x)\}$ we have $i(w; f) \geq 0$. So by using theorem A.10 we see that

$$\begin{aligned} r &= i(x; f) \\ &= \mu\left(f(x), f, U\right) \\ &= \sum_{w \in U \cap f^{-1}\{f(x)\}} i(w, f) \\ &\geq i(x; f) \\ &= r. \end{aligned}$$

This can only happen when $\sharp(U \cap f^{-1}\{f(x)\}) = 1$, which means that only x is mapped to $f(x)$. As x was arbitrary, $f|_{U \cap F}$ is injective.

□

Note that

$$B_f = \{x \in \mathcal{M} : |i(x; f)| \neq 1\}.$$

We state the following basic result of topological index which follows from the previous theorem and the fact that the index is locally constant (± 1) outside the branch set.

Lemma A.14. *The function $\mathcal{M} \to \mathbb{Z}, x \mapsto i(x; f)$ is upper semicontinuous.*

B Open problems

Conjecture B.1. *Let X be a metric space. If there exists a net in X with at most exponential growth rate, then $\mathrm{Ord}_*(X)$ contains a minimal element.*

Conjecture B.2. *Let $\mathcal{M}$ be a length manifold and assume $f\colon \mathbb{R}^n \to \mathcal{M}$ is a BLD mapping. Then $\mathcal{M}$ has a diameter bounded fundamental group.*

Conjecture B.3. *Let $\mathcal{M}$ be a length manifold and assume $f\colon \mathbb{R}^n \to \mathcal{M}$ is a BLD mapping. Then the set of homotopyclasses of geodesic triangles in $\mathcal{M}$ is finite.*

If $z \in p_{\mathcal{M}}^{-1}\{x_0\}$, we denote

$$p_n^{x_0} := \{\alpha_1 * \cdots * \alpha_n \mid \alpha_j\colon x_j \underset{g}{\curvearrowright} x_{j+1}, x_j \in \mathcal{M}, x_1 = x_n = x_0\},$$
$$\widetilde{p}_n^{z} := \{\widetilde{\alpha} \mid \alpha \in p_n^{x_0}, \widetilde{\alpha}(0) = z\}$$

and

$$\mathcal{P}_n^{x_0} := \{[\omega] \in \Pi_1(\mathcal{M}, x_0) \mid \omega \in p_n^{x_0}\}, \quad \widetilde{\mathcal{P}}_n^{z} := \{[\widetilde{\omega}] \mid \omega \in \widetilde{p}_n^{x_0}\}$$

An open-ended idea B.4. *We have a chain of inclusions*

$$\emptyset = \mathcal{P}_0^{x_0} \subset \mathcal{P}_1^{x_0} \subset \mathcal{P}_2^{x_0} \subset \ldots,$$

and actually even $\Pi_1(\mathcal{M}, x_0) = \bigcup_{n\in\mathbb{N}} \mathcal{P}_n^{x_0}$.

One could try to construct some sort of chain complexes (or graded algebras) of these, either by taking $A_n := \mathcal{P}_{n+1}^{x_0} - \mathcal{P}_n^{x_0}$ or $B_n := \mathcal{P}_{n+1}^{x_0} \Big/ \mathcal{P}_n^{x_0}$.

In both cases the mapping $\mathcal{K}$ induced by path composition gives us a mapping

$$\mathcal{K}\colon \Pi_1(\mathcal{M}, x_0) \times \Pi_1(\mathcal{M}, x_0) \to \Pi_1(\mathcal{M}, x_0)$$

such that for each $m, n \in \mathbb{N}$ we have that

$$\mathcal{K}\left[\mathcal{P}_n^{x_0} \times \mathcal{P}_m^{x_0}\right] \subset \bigcup_{j=1}^{m+n} \mathcal{P}_j^{x_0}.$$

This resembles in some sense the behaviour of the wedge product in the graded algebra formed by k-forms in differential geometry.

Some sort of exterior derivative could also be defined in the lines of

$$d(\alpha_1 * \cdots * \alpha_k) = \alpha_1 * \cdots * \hat{\alpha}_k,$$

where α_i are geodesics defining a geodesical k-polygon, and $\hat{\alpha}_k$ is a path $\gamma * \gamma'$, where $\gamma\colon \alpha_k(0) \underset{g}{\curvearrowright} x$ and $\gamma'\colon x \underset{g}{\curvearrowright} x_0$. ($x \in \mathcal{M}$)

(Or by trying something more fancy like

$$d(\alpha_1 * \cdots * \alpha_k) = \text{"}\sum_{j=1}^{k}\text{"}\alpha_1 * \cdots \hat{\alpha}_j * \cdots * \alpha_k,$$

where α_i are geodesics defining a geodesical k-polygon, and $\hat{\alpha}_j$ is a path $\gamma * \gamma'$, where $\gamma\colon \alpha_j(0) \underset{g}{\curvearrowright} x_0$ and $\gamma'\colon x_0 \underset{g}{\curvearrowright} \alpha_j(1)$. *)*

Conjecture B.5. *Let $\mathcal{M}$ be a length manifold and assume $f\colon \mathbb{R}^n \to \mathcal{M}$ is a BLD mapping. Then $\sharp\mathcal{P}^n_{x_0} < \infty$.*

Conjecture B.6. *Let $\mathcal{M}$ be a length manifold and assume $\sharp\mathcal{P}^n_{x_0} < \infty$ for some $n \in \mathbb{N}$. Then $\sharp\mathcal{P}^k_{x_0} < \infty$ for* all *k.*

Conjecture B.7. *Let X and Y be two metric spaces. Suppose there exists coarse Lipschitz quotient mappings $X \to Y$ and $Y \to X$. Then there exists a coarse quasi-isometry $X \simeq Y$.*

An open-ended idea B.8. *The growth rate does not contain information about how many unbounded directions a space has. ($\mathrm{Ord}(\mathbb{N}) = \mathrm{Ord}(\mathbb{Z})$) On the other hand a coarse quasi-isometry is able to see such things. ($\mathbb{N}$ is not CQI with $\mathbb{Z}$) Could one introduce a more strict version of growth rate that would say something like "same growth rates imply existence of CQI".*

Or could one measure the coarse difference between spaces with the same growth rate by doing some (co)homology theory by constructing suitable chain complexes? For example: The spaces

$$\begin{aligned}&\mathbb{R}^2,\\ &\mathbb{R}^2_+ = \mathbb{R} \times [0,\infty[\text{ and}\\ &\mathbb{R}^2_{++} = \mathbb{R} \times [0,\infty[\cap[0,\infty[\times\mathbb{R}\end{aligned}$$

all have the same growth rate $\mathcal{O}(x^2)$. There is no coarse quasi-isometries between any of these, but we do have regular mappings

$$\mathbb{R}^2 \to \mathbb{R}^2_+ \to \mathbb{R}^2_{++}$$

and inclusions on the opposite direction.

An open-ended idea B.9. *Could one take an axiomatic approach to growth (rate) ?*

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