Interactions between quasiregular- and BLD-mappings

Rami Luisto

Academic dissertation

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List of included articles

This dissertation consists of an introductory part and the following publications:

[A] R. Luisto and P. Pankka - Rigidity of Extremal Quasiregularly elliptic manifolds. (Accepted for publication in *Groups, Geometry & Dynamics.*)
[http://arxiv.org/abs/1307.7940]

[B] R. Luisto - Note on Local to Global properties of BLD-mappings. *Proceedings of the AMS*, **144** (2016), 599-607.

[C] R. Luisto - A Characterization theorem for BLD-mappings.
[http://arxiv.org/abs/1509.01832]

In this introductory part these articles will be referred to as [A], [B] and [C]. The paper [A] contains original and meaningful contributions by the author while the articles [B] and [C] consist of the author's independent research.

Contents

Introduction

We study the theories of *quasiregular mappings* and *mappings of Bounded Length Distortion* – QR- and BLD-mappings for short. A connecting element of the papers [A], [B] and [C] is the interplay of these two classes of mappings. More precisely, the main ideas in paper [A] were motivated by results concerning BLD-mappings and expanding the BLD-results to the quasiregular setting in a weaker form. On the other hand, the idea for the paper [B] came as the author was proving for BLD-mappings in metric spaces a result known for quasiregular mappings between Riemannian manifolds. Finally the results of [C] arose from the interest to understand which properties of BLD-mappings rely on certain Euclidean tools that are used with quasiregular mappings between Euclidean domains, but are no longer applicable in more general path-metric spaces.

In the first part of this summary we give definitions of these classes of mappings and discuss their historical origin. After this we study the ideas and results in articles [A], [B] and [C].

1.1 Quasiregular- and BLD-mappings

A non-constant continuous mapping $f\colon M \to N$, where M and N are oriented complete Riemannian n-manifolds, is K-*quasiregular* if f belongs to the Sobolev space $W^{1,n}_{\mathrm{loc}}(M,N)$ and satisfies the distortion inequality

$$\|Df\|^n \le K J_f \quad \text{a.e. in } M, \tag{QR}$$

where Df is the differential of the map f and J_f the Jacobian determinant. The assumption on Sobolev regularity guarantees the existence of weak derivatives almost everywhere, so that we may meaningfully state the condition (QR). For the definition of the Sobolev space $W^{1,n}_{\mathrm{loc}}(M,N)$ see e.g. [38, Section 4.2]. A homeomorphic quasiregular mapping is called a *quasiconformal* mapping. By Reshetnyak's Theorem, quasiregular mappings are *branched covers*, that is, discrete and open mappings, see e.g. [77, Theorem I.4.1].

The study of spatial quasiconformal maps was initiated by Gehring, [17, 18, 19, 20, 21, 22, 23]. Two-dimensional quasiregular maps were first mentioned by

Grötzsch in [27] and in the higher dimensions their definition was suggested by Lavrentiev, [45]. Their systematic study, however, is due to Yu. G. Reshetnyak in a series of articles published in the mid sixties and -seventies; [64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74]. The geometric theory was then advanced by Martio, Rickman and Väisälä in a sequence of papers [50, 51, 52] and in the book by Rickman [77]; see also Bojarski and Iwaniec [5].

Given $L \geq 1$, a branched cover $f\colon X \to Y$ between metric spaces is a mapping of *Bounded Length Distortion*, or (L-)BLD for short, if f satisfies the inequality

$$L^{-1}\ell(\beta) \leq \ell(f \circ \beta) \leq L\ell(\beta). \tag{BLD}$$

for all paths $\beta\colon [0,1] \to X$. Here $\ell(\beta) \in [0,\infty]$ is the length of a path $\beta\colon [0,1] \to X$ defined by

$$\ell(\beta) := \sup \left\{ \sum_{j=1}^{N} d(\beta(t_{j-1}), \beta(t_j)) \mid 0 = t_0 \leq \ldots \leq t_N = 1, N \in \mathbb{N} \right\}.$$

Paths with finite or infinite length are called *rectifiable* or *unrectifiable*, respectively.

BLD-mappings were first introduced by Martio and Väisälä in [53] as a subclass of quasiregular mappings between Euclidean domains; every L-BLD-mapping is $K(L)$-quasiregular. Since their introduction, BLD-mappings have been used extensively in the contemporary study of geometric analysis and geometric topology, see e.g. [14, 30, 31, 32, 34, 35, 36, 46, 47, 60]. As noted already by Martio and Väisälä, BLD-mappings admit several equivalent definitions in the Euclidean setting. The metric theory of these equivalent definitions is the subject of both the paper [C] and Section 4 of this summary. See also [29] for another characterization of BLD-mappings in the Euclidean setting.

We give next a few standard examples of BLD- and quasiregular mappings. First of all, for any closed and oriented Riemannian manifold N a covering map $p\colon \widetilde{N} \to N$ is a smooth local isometry and as such both a 1-quasiregular and a 1-BLD-mapping. More generally any smooth local isometry between complete and oriented Riemannian n-manifolds is both a 1-quasiregular and a 1-BLD-mapping.

Secondly, *the winding map* $w\colon \mathbb{R}^2 \to \mathbb{R}^2$ is the mapping defined in polar coordinates as $(r,\varphi) \mapsto (r, 2\varphi)$, or as

$$w\colon \mathbb{C} \to \mathbb{C}, \quad z \mapsto \frac{z^2}{|z|}$$

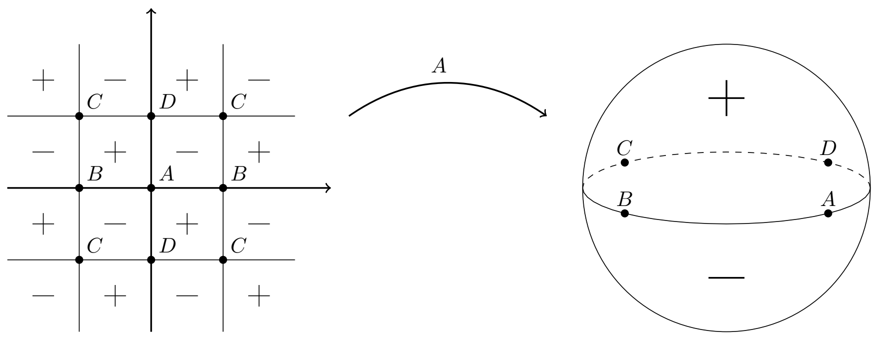

Figure 1: The Alexander map.

in the complex notation. The winding map is not holomorphic but both a BLD-mapping and a quasiregular mapping. Since direct products of L-BLD-mappings are L-BLD, we note that the higher dimensional analogues,

$$W\colon \mathbb{R}^2 \times \mathbb{R}^{n-2} \to \mathbb{R}^n, \quad W = w \times \mathrm{id}_{\mathbb{R}^{n-2}}$$

are also both BLD- and quasiregular mappings.

The next example is the *Alexander map* $A\colon \mathbb{R}^2 \to \mathbb{S}^2$, see Figure 1. We define the mapping by first tiling the Euclidean plane with translations of the unit square and labeling the tiles and their corners as in the figure. We fix two bilipschitz mappings f_+ and f_- that take the unit square to the upper or lower hemisphere of $\mathbb{S}^2$, agree on the boundary of the unit square and map the boundary of the unit square to the equator of the sphere with the marked points A, B, C and D mapped to the corresponding marked points in the sphere. The Alexander map is then defined by mapping the Euclidean plane to the sphere via the affine translations of these mappings as per the labeling of the tiling. The Alexander map is, again, both a BLD- and a quasiregular mapping. Note that outside the labeled points the mapping resembles a covering map while at these points the mapping is locally similar to the winding map w. The Alexander map naturally factors through the torus, c.f. the Weierstrass $\wp$-function [2, Section 3.1].

Our final example is the famous *Zorich map* $Z\colon \mathbb{R}^3 \to \mathbb{R}^3$, see [83], which we define via the Alexander map by setting

$$Z\colon \mathbb{R}^3 \to \mathbb{R}^3, \quad Z(x, y, z) = e^z A(x, y).$$

The Zorich map is a quasiregular mapping but not a BLD-mapping since for $z \in \mathbb{R}$ the length distortion of paths lying in a horizontal plane

$$T_z := \{(x, y, z) \in \mathbb{R}^3 \mid x, y \in \mathbb{R}\}$$

is comparable to e^z. Note that the Zorich map is not surjective and similar examples of non-surjective quasiregular mappings exist in all dimensions $n \geq 2$; see also [14] and [13].

1.2 From Riemannian manifolds to metric spaces

The study of quasiregular- and BLD-mappings arose from the theory of holomorphic mappings of one complex variable, and originally both quasiregular and BLD-mappings were defined between Euclidean domains. Since their definitions are analytical, it makes sense that their theory is viable also in the setting of Riemannian manifolds – especially with some extra assumptions on bounded curvature which induce Ahlfors regularity for the spaces in question. (See e.g. [40, Lemma 2.3].)

On the other hand the definition of BLD-mappings is metric and we may ask if they have a reasonable theory when defined between the generality of (path)metric spaces. This is, indeed, one of the main topics of [C] in which we study BLD-mappings between spaces that are *locally compact* and *complete* path-metric spaces. The assumption of a path-metric is natural considering the definition of BLD-mappings, although quasiconvexity is a good possible alternative. The local compactness and completeness are required to guarantee the existence of so called normal neighbourhoods and lifts of paths.

Between these two extremes – complete Riemannian manifolds with curvature bounds and locally compact and complete path-metric spaces – there is another natural class of spaces where the theory appears naturally. As in [35, Definition 1.1] we say that X is a *generalized n-manifold* if it is a locally compact, connected and locally connected Hausdorff space of finite topological dimension such that

(GM-1) X has *cohomological dimension* $\dim_{\mathbb{Z}} X$ of at most n, i.e. $H^p_c(U) = 0$ for all open $U \subset X$ and $p \geq n+1$, and

(GM-2) for each $x \in X$ and each open neighbourhood U of x there exists another open neighbourhood V of x contained in U for which

$$H^p_c(V) = \begin{cases} \mathbb{Z}, & \text{for } p = n, \\ 0, & \text{for } p = n-1 \end{cases}$$

and such that the standard homomorphism $H_c^n(W) \to H_c^n(V)$ is surjective for all neighbourhoods W of x in V.

Here $H_c^p(\cdot)$ is the compactly supported Alexander-Spanier cohomology, see [59, Section 3] or [54]. Topological manifolds are always generalized manifolds, but not vice versa, see e.g. [35, Example 1.4.(a)]. The motivation behind the definition of generalized manifolds is that it is essentially the minimal setting in which the *topological index- and degree theory* is applicable for branched covers; see, again, [59, Section 3]. This, in turn gives rise to topological equidistribution results for quasiregular- and BLD-mappings. Besides this topological assumption it is customary to define, in this context, the following conditions that a metric space X may or may not satisfy:

(A1) X is n-rectifiable and has locally finite Hausdorff n-measure,

(A2) X is locally Ahlfors n-regular,

(A3) X is locally bi-Lipschitz embeddable in Euclidean space, and

(A4) X is locally linearly contractible.

In [35, Section 5] generalized manifolds satisfying the conditions (A1)-(A4) are called *generalized manifolds of type A* and described as *"a class of generalized manifolds whose Lipschitz analysis is similar to that on Riemannian manifolds (cf. [36])"*. A generalized manifold of type A has the topological structure to enable the strong local covering properties that quasiregular mappings have in the Euclidean setting, while the type A -requirement enables the use of results in Lipschitz analysis. Thus despite its technical definition this class of spaces forms a natural setting for the study of quasiregular- and BLD-mappings. There are several open questions about the necessity of all properties (A1)-(A4) for the theory of quasiregular- and BLD-mappings. To mention one of them we note that by [35, Theorem 6.4] for a BLD-mapping between generalized n-manifolds of type A the branch set has zero measure. Heinonen and Rickman state that: *"We do not know whether Theorem 6.4 remains valid for BLD-maps between more general spaces."* The result [35, Theorem 6.4] is, indeed, the only reason we prove certain theorems in [C] in the setting of generalized manifolds of type A instead of lesser requirements.

1.3 From holomorphic mappings to BLD-theory

In this section we find a motivation for the definitions of quasiregular- and BLD-mappings by approaching them as generalizations of holomorphic maps.

Historically the theory of quasiregular mappings has its roots in complex analysis – for an extensive survey of the history of quasiregular mappings see e.g. [81], and for quasiconformal mappings e.g. [38]. There are various ways of generalizing holomorphic maps. Here we take the geometric approach of studying how the tangent map of a holomorphic map distorts the unit ball. Another approach would be to generalize the Cauchy-Riemann equations – this leads to the study of the Beltrami equation, see e.g. [38] and [7].

Let us consider a holomorphic mapping

$$f\colon \mathbb{C} \to \mathbb{C}, \quad z \mapsto (u(z), v(z)).$$

Holomorphic mappings are everywhere differentiable, so especially the matrix of partial differentials of f exists at every point. By applying the Cauchy-Riemann equations at a point $(x_0, y_0) \in \mathbb{R}^2$ and denoting

$$\alpha := \partial_1 u(x_0, y_0) \quad \text{and} \quad \beta := \partial_2 u(x_0, y_0)$$

we may write

$$Df(x_0, y_0) = \begin{bmatrix} \partial_1 u & \partial_2 u \\ \partial_1 v & \partial_2 v \end{bmatrix} =: \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}.$$

Thus, for the Jacobian of f at (x_0, y_0), we have

$$J_f(x_0, y_0) := \det[Df(x_0, y_0)] = \alpha^2 + \beta^2 \geq 0$$

and by denoting $c^2 = J_f(x_0, y_0)$, we observe by a direct calculation that for any $\mathbf{w} \in \mathbb{R}^2$,

$$\|(Df(x_0, y_0))\mathbf{w}\| = c\|\mathbf{w}\|. \tag{1.1}$$

Thus for the tangent map of a holomorphic mapping at any point, balls of radius r get mapped to balls of radius cr. Another way to state this is that holomorphic maps are *conformal mappings*, in particular the square of the operator norm of the differential equals the Jacobian:

$$\|Df(z)\|^2 = J_f(z). \tag{C}$$

By interpreting balls of the tangent space as "infinitesimal balls", we can state this fundamental property of holomorphic mappings heuristically as follows:

Infinitesimally, holomorphic mappings map balls to balls.

Both this geometric heuristic definition and the conformality property lend themselves when generalizing holomorphic mappings. Holomorphic mappings are defined in the complex domain, and the first approach to generalizations is to extend the definition to n-dimensional Euclidean domains via the conformality condition. It turns out, however, that conformal mappings between Euclidean spaces of higher dimensions form a very restricted collection of mappings. Indeed, by the Liouville theorem such mappings are just the so called Möbius transformations, see e.g. [38, Chapter 5].

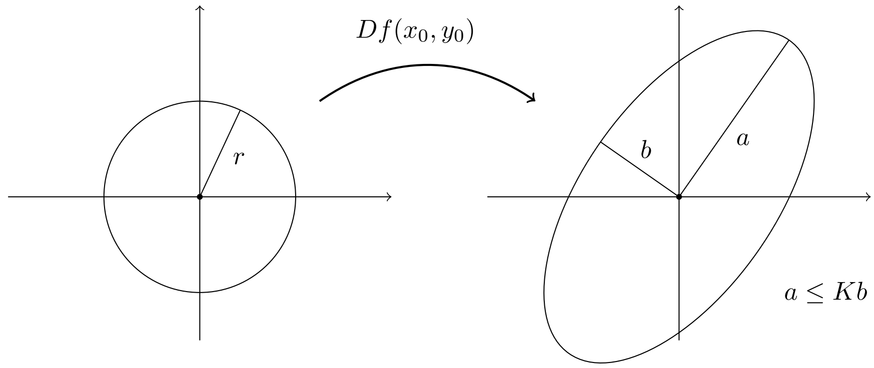

Figure 2: The canonical picture describing quasiregular mappings via the behaviour of their tangent maps.

Instead of using the conformal equality (C) we turn to the quasiregular inequality (QR) stated earlier; under the weak differentiability requirements we require that

$$\|Df(\mathbf{x})\|^n \leq K J_f(\mathbf{x})$$

holds for almost every $\mathbf{x} \in \mathbb{R}^n$. This is clearly a proper generalization of the conformality property (C) and just like the conformality property the condition (QR) has a heuristic geometric interpretation in terms of infinitesimal balls. To see this we look at the differential of a quasiregular mapping at a point $\mathbf{x} \in \mathbb{R}^n$ and note that as a (non-degenerate) linear mapping the differential maps a unit sphere to an ellipsoid, see Figure 2. Furthermore the operator norm $\|Df(\mathbf{x})\|$ of the differential is the length of the major axis of said ellipsoid, while the Jacobian is, up to constant depending on n, the volume of the ellipsoid. The connection between the minor- and major axes of an ellipsoid to its volume follows by approximating the volume of an ellipsoid from above and below by the respective smallest and largest balls contained in- and containing the ellipsoid, see Figure 3. Thus the geometric interpretation of the inequality (QR) is that quasiregular mappings map infinitesimal balls into ellipsoids that have the n:th power of the length of their major axis comparable to their volume. But a moment's thought shows that this is possible if and only if the minor axis of the ellipsoid is comparable to the major axis. We denote

$$\ell(Df(\mathbf{x})) := \min_{w \in \mathbb{S}^{n-1}} \|Df(\mathbf{x})w\|$$

and see that for quasiregular mappings we have both

$$\ell(Df(\mathbf{x})) \leq \|Df(\mathbf{x})\| \leq K\ell(Df(\mathbf{x}))$$

and

$$0 < J_f(\mathbf{x}) \leq \|Df(\mathbf{x})\|^n \leq KJ_f\mathbf{x} < \infty$$

at almost every $\mathbf{x} \in \mathbb{R}^n$. This means that for quasiregular mappings the minimal- and maximal stretchings of the tangent map at any point are comparable to each other and the volume of the image of the unit ball in the tangent plane. (Up to the correct exponent.) Thus we can heuristically give the (QR) condition as follows:

> *Infinitesimally, quasiregular mappings map balls into ellipsoids with uniformly controlled eccentricity.*

See, again, Figure 2 for a the canonical picture used for this definition. The term "controlled eccentricity" in the heuristic geometric definition of quasiregular mappings can be interpreted in many ways, one example being the uniform constant bounding the ratio of the major and minor axes. To give another natural example, a different interpretation gives rise to *mappings of finite distortion*, which have been studied by several authors in recent years, see e.g. [3] and [37].

The geometric interpretation of the definition of quasiregular mappings emerges in studying the structure of the image of an infinitesimal ball. In our approach we have focused on the shape of the image whilst ignoring the diameter. For holomorphic mappings we see again from (1.1) that the diameter of an infinitesimal ball is scaled by the square root of the Jacobian. The scaling factor of a holomorphic map can take arbitrarily small or large values in $(0, \infty)$, e.g.

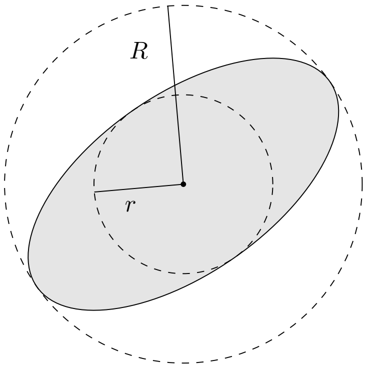

Figure 3: The volume of an ellipsoid is controlled by the volume of the balls with radii of equaling the minor- and major axes of the ellipsoid, $\pi r^2 \leq \mathrm{vol}(E) \leq \pi R^2$.

for the complex exponential we have $J_{\exp}(z) = e^{2\,\mathrm{Re}(z)}$. Similarly for general quasiregular mappings the images of infinitesimal unit balls can have arbitrarily large or small volume, c.f. the Zorich map.

It is then quite natural to ask what class of mappings emerges by constricting the size of the Jacobian of K-quasiregular mappings. A straightforward requirement is to ask for uniform lower and upper bounds for the Jacobian. As noted earlier, for a quasiregular mapping the Jacobian is comparable, up to an exponent, to the the minimal- and maximal stretchings of the tangent map. Thus a uniform restriction to the Jacobian gives bounds also to the minimal- and maximal stretchings of the tangent map; these bounds in turn lead to the following heuristic definition.

> *Infinitesimally, BLD-mappings map balls into ellipsoids with a bounded eccentricity and diameter.*

Note that BLD-mappings have no direct correspondent in the theory of (entire) holomorphic mappings, since a holomorphic entire mapping with a bounded Jacobian is a complex linear function by Liouville's theorem. In this respect it is then almost surprising that the class of BLD-mappings is an interesting and rich collection of maps.

Using this heuristic idea Martio and Väisälä give in [53] one definition of BLD-mappings as quasiregular mappings with

$$L^{-1}\|h\| \leq \|Df(h)\| \leq L\|h\| \tag{BLD*}$$

for all $h \in \mathbb{S}^{n-1}$ at almost all points of the domain. This analytical definition is one of the motivations for the definition of L-radial mappings discussed in Section 4.

Besides this analytical definition given by Martio and Väisälä it was shown that the analytical definition of BLD-mappings is equivalent to other definitions with more metric flavor, [53, Theorem 2.16]. These definitions arise naturally from noting that by the analytical BLD-condition (BLD*) the image of a ball $B(0,r)$ in the tangent space must contain and be contained in balls of radius $L^{-1}r$ and Lr, respectively. Since the differential of a mapping approximates the original mapping in small scales we arrive to the following heuristical definition of BLD-mappings, see Figure 4.

> *BLD-mappings map small balls onto sets that resemble the original ball in a metrically bounded fashion.*

As with the term "controlled eccentricity" in the heuristical definition of quasiregular mappings, the term "metrically bounded fashion" lends itself to a plethora of interpretations, which we will study later in Section 4.

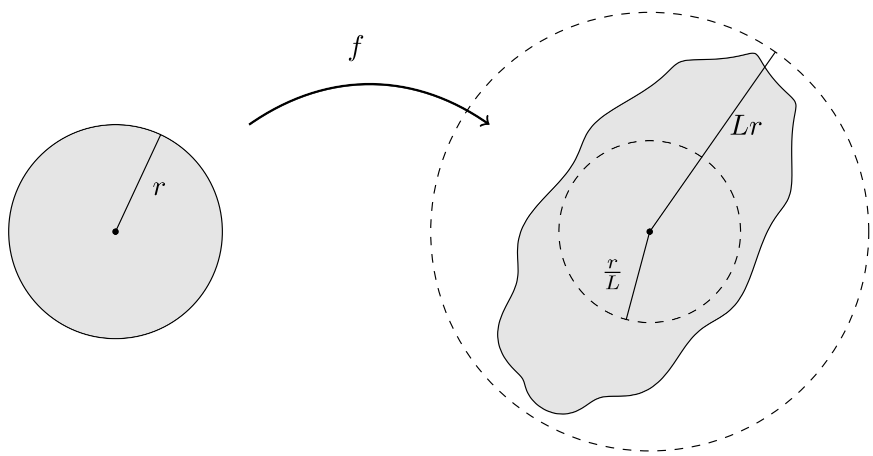

Figure 4: A picture describing the local definition of BLD-mappings.

2 Quasiregular- and BLD-ellipticity

In this section we discuss quasiregular ellipticity results related to paper [A]. Our approach revolves around the idea of how to translate ideas from BLD-theory for quasiregular mappings and on the interplay of the geometry of groups and spaces.

2.1 Elliptic spaces

A basic question in the theory of quasiregular mappings in higher dimensions is that of quasiregular ellipticity. A closed Riemannian n-manifold N is said to be *quasiregularly elliptic* if there exists a non-constant quasiregular mapping $f\colon \mathbb{R}^n \to N$. (This terminology was first coined in [6] and suggested from the discussion in [26, pp. 63–67].) The question about which closed and oriented Riemannian manifolds are quasiregularly elliptic has be studied extensively, to note some of the first works in this area we mention [76], [39] and [61].

The examples of quasiregular mappings in Section 1.1 immediately give rise to two examples of quasiregularly elliptic manifolds. Since covering maps are quasiregular, the n-torus $\mathbb{T}^n$ is quasiregularly elliptic via the covering map

$$p\colon \mathbb{R}^n \to \mathbb{S}^1 \times \cdots \times \mathbb{S}^1, \qquad (t_1, \ldots, t_n) \mapsto ((\cos t_1, \sin t_1), \ldots, (\cos t_n, \sin t_n)) .$$

Also the unit sphere $\mathbb{S}^2$ is quasiregularly elliptic via the existence of the Alexander map. In higher dimensions similar constructions yield the quasiregular ellipticity of $\mathbb{S}^n$. Thus the torus and the sphere are always quasiregularly elliptic and in dimension $n = 2$, these are the only quasiregularly elliptic manifolds by the uniformization theorem ([63], [42, 43, 44]) and Stoilow factorization ([79, p.120]). Since direct products of BLD-mappings are still BLD we get more examples of quasiregularly elliptic manifolds, namely, the direct products of spheres and tori. Indeed, in dimension $n = 3$, the possible closed targets of quasiregular mappings from $\mathbb{R}^3$ are $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{S}^1$, $\mathbb{T}^3$, and their quotients. The completeness of this list follows from the geometrization theorem; see Jormakka [39]. In dimensions $n \geq 4$ there are more complicated examples of quasiregularly elliptic manifolds, e.g. the join $(\mathbb{S}^2 \times \mathbb{S}^2)\#(\mathbb{S}^2 \times \mathbb{S}^2)$ is a quasiregularly elliptic by [78]. In dimensions $n \geq 4$ the question which closed and oriented Riemannian manifolds are quasiregularly elliptic is widely open. For example the following question is due to Gromov, [25, p. 200]: *Are all simply connected closed* 4-*manifolds quasiregularly elliptic?* The classification of all quasiregularly elliptic closed manifolds in dimensions $n \geq 4$ is a formidable problem that might not have a simple solution. There are, however, many results giving restrictions to the geometry of quasiregularly elliptic closed manifolds.

In higher dimensions the basic obstructions to quasiregular ellipticity arise from the geometric structure of the fundamental group of the manifold in question. (See, however, [6] for the groundbreaking results in higher (co)homology-groups by Bonk and Heinonen.) These classical restrictions are given via concepts relating to the growth rate of groups, which we define next. In any finitely generated group we may define a *word metric* via the length of paths in the group's Cayley graph, see e.g. [8, Section 3.2.3] for the basic definitions. We are interested in the "volume" of large balls of the group. Finitely generated groups equipped with the word metric are discrete metric spaces and all balls have finitely many points. More precisely, in any finitely generated group G with a finite generating set S we have the following bound

$$\#\overline{B}(e_G, r) \leq (\#S)^r$$

for $r > 0$. We say that a finitely generated group G has *at most polynomial growth rate* if there exists constants $C \geq 1$ and $Q > 0$ such that

$$\#\overline{B}(e_G, r) \leq Cr^Q$$

for all $r > 0$. We denote this as $\mathrm{ord}(G) \leq Q$. By the Bass formula [4], if a finitely generated group G has at most polynomial growth rate, then there exists constants $C \geq 1$ and $n \in \mathbb{N}$ such that

$$C^{-1}r^n \leq \#\overline{B}(e_G, r) \leq Cr^n.$$

In this case we write $\mathrm{ord}(G) = n$ and say that the group has polynomial growth rate of degree n.

Finally, we say that a group G *is virtually (P)*, if there exists a subgroup $H \leq G$ of finite index such that H is (P). The fact that virtually nilpotent groups have polynomial growth rate was originally proven by Milnor and Wolf, [55, 82]. A group G is said to be *nilpotent* if it has a finite *central series*, i.e. there exists a sequence of subgroups

$$\{e_G\} = A_0 \triangleleft A_1 \triangleleft \cdots \triangleleft A_n = G$$

with $[G, A_j] \leq A_{j-1}$ for all $j = 1, \ldots, n$. The property of a group being nilpotent is sometimes heuristically described as a property of being "almost abelian". For an *abelian* group G with a finite generating set S the Milnor-Wolf theorem is easy to prove; a simple counting argument shows that for an abelian group G

$$\#\overline{B}(e_G, r) \leq 2(r+1)^Q \leq Cr^Q,$$

where Q is the size of the generating set S. A celebrated result of Gromov, [24], shows that the converse of the Milnor-Wolf theorem is also true and thus connects the growth rate of a group to its algebraic properties:

Theorem (Gromov)**.** *Let G be a finitely generated group. The group G has polynomial growth rate if and only if it is virtually nilpotent.*

The basic result connecting quasiregular ellipticity to the growth rate of groups is the following Varopoulos's Theorem, [57].

Theorem (Varopolous)**.** *Let N be a closed quasiregularly elliptic n-manifold. Then the fundamental group $\pi_1(N)$ has at most polynomial growth rate.*

From this result we can already deduce that oriented surfaces with genus at least 2 are not quasiregularly elliptic – their fundamental groups have exponential growth rate, see also [39]. Combined with Gromov's Theorem, Varopoulos's result yields that *the fundamental group of a quasiregularly elliptic manifold is in fact virtually nilpotent.* However in all known examples the fundamental group of a quasiregularly elliptic manifold is virtually *abelian.* If the class of quasiregular mappings is replaced by BLD-mappings the fundamental group indeed is virtually abelian – this result is due to Le Donne and Pankka, [47]. Note that all known closed quasiregularly elliptic manifolds are also BLD-elliptic. For quasiregular mappings it is an open question if the fundamental group of a quasiregularly elliptic manifold is always abelian. Article [A] gives a partial answer to this question. In [A] our main result is the following.

Theorem 1 ([A, Theorem 1.1])**.** *Let N be a closed quasiregularly elliptic n-manifold. Then the following are equivalent:*

1. $\mathrm{ord}(\pi_1(N)) = n$,
2. *N is aspherical, and*
3. *$\pi_1(N)$ is virtually $\mathbb{Z}^n$ and torsion free.*

Recall that a manifold N is aspherical if its universal cover $\widetilde{N}$ is contractible. Note that since the group $\mathbb{Z}^n$ is an abelian group, Theorem 1 immediately yields that the fundamental group of an extremal (in the sense of (1)) quasiregularly elliptic manifold is virtually abelian.

In Theorem 1 the implication (3) $\Rightarrow$ (1) is trivial, and the implication (2) $\Rightarrow$ (3) follows from strong results in cohomological group theory. Thus the main content of [A] is in the proof of implication (1) $\Rightarrow$ (2). A crucial observation is to note that the asphericality of N is implied by the existence of a *proper* branched cover $\mathbb{R}^n \to \widetilde{N}$, where $\widetilde{N}$ is the universal cover of N. This is formulated as the following lemma.

Lemma 2 ([A, Lemma 3.2.]). *Let N be a compact n-manifold and $f\colon \mathbb{R}^n \to \widetilde{N}$ a proper branched cover onto the universal cover $\widetilde{N}$ of N. Then N is aspherical.*

Recall that a *proper map* is a continuous map for which the pre-images of compact sets are compact. In the setting of the paper [A], a quasiregular mapping $f\colon \mathbb{R}^n \to \widetilde{N}$ is proper if and only if it is *finite-to-one*, i.e. if there exists a constant m for which $\#f^{-1}\{y\} \leq m$ for all $y \in \widetilde{N}$.

Here is where the BLD-intuition mentioned in the introduction appears. For a BLD-mapping $f\colon \mathbb{R}^n \to N$, where N is a closed n-manifold that satisfies (1), the lift $\tilde{f}\colon \mathbb{R}^n \to \widetilde{N}$ is a finite-to-one mapping – this follows from an elementary volume counting argument and the fact that the spaces $\widetilde{N}$ and $\pi_1(N)$ are coarsely quasi-isometric. For general quasiregular mappings this does not hold true. We may, for example, combine the Zorich map $Z\colon \mathbb{R}^3 \to \mathbb{R}^3$ with the covering map $p\colon \mathbb{R}^3 \to \mathbb{T}^3$ to obtain a non-constant quasiregular mapping $f\colon \mathbb{R}^3 \to \mathbb{T}^3$ onto a oriented and closed Riemannian manifold with the lift $\tilde{f} = Z\colon \mathbb{R}^3 \to \mathbb{R}^3$ being infinite-to-one. To avoid this problem we reduce the question to polynomial quasiregular mappings.

2.2 Polynomial quasiregular mappings

Let $\widetilde{N}$ be the universal cover of a closed and oriented Riemannian manifold. A quasiregular mapping $f\colon \mathbb{R}^n \to N$ is a polynomial quasiregular mapping if the measure

$$\mu(A) := \int_A J_f$$

is doubling. We refer to [59] for the definition and basic results concerning polynomial quasiregular mappings; see also [58] and [60] for an other subclass of quasiregular mappings defined via a growth condition. Note that polynomial quasiregular maps contain complex polynomials, but not the complex exponential map. In higher dimensions e.g. BLD-mappings are are always polynomial quasiregular since the Jacobian is uniformly bounded, but the Zorich map is not.

The class of polynomial quasiregular mappings shares many properties of BLD-mappings. Indeed in [A] we rely on a characterization of Onninen and

Rajala, [59, Theorem 12.1], which implies that a polynomial quasiregular mapping $f\colon \mathbb{R}^n \to X$ to an *n-Loewner* and *Ahlfors n-regular* oriented and complete Riemannian n-manifold is finite-to-one. We discuss the Loewner condition later on, the Ahlfors n-condition states just that there exists a constant $C \geq 1$ such that

$$C^{-1} r^n \leq \mathcal{H}^n(B(x,r)) \leq C r^n$$

for all $x \in X$ and $r > 0$, where $\mathcal{H}^n$ is the Hausdorff n-measure of X. Thus the class of polynomial quasiregular mappings has a similar finite-to-one -property as BLD-mappings do.

A crucial step used in [A] is then to replace a general quasiregular mapping $f\colon \mathbb{R}^n \to N$ by a polynomial quasiregular mapping. But this is the Miniowicz-Zalcman Lemma of Bonk and Heinonen, [6, Corollary 2.2]. Indeed, by their results, if there exists a non-constant quasiregular mapping $f\colon \mathbb{R}^n \to N$, where N is an closed and oriented quasiregular mapping, then there also exists a non-constant uniformly Hölder continuous quasiregular mapping $f\colon \mathbb{R}^n \to N$ that satisfies

$$\int_{B(x,r)} J_f \leq C r^n, \tag{H}$$

for $r \geq 1$, where C is a constant independent of the ball $B(x,r)$. Since the volume of balls in $\mathbb{R}^n$ is roughly r^n, this condition means heuristically that the Jacobian J_f is bounded by C on average in balls of large diameter. As is noted in [A], in the setting where $f\colon \mathbb{R}^n \to X$ is a non-constant quasiregular mapping into a complete Riemannian manifold that is both Ahlfors n-regular and n-Loewner, the condition (H) actually implies that f is a polynomial quasiregular mapping. Note, however, that the condition (H) is a strictly stronger condition; for example the complex polynomial $z \mapsto z^k$, $k \geq 2$, is a polynomial quasiregular mapping, but (H) does not hold without the exponent being at least $k + 1 > n = 2$.

2.3 The n-Ahlfors and n-Loewner conditions of $\widetilde{N}$

Since in the setting of Theorem 1 the quasiregular mapping $f\colon \mathbb{R}^n \to N$ may be replaced by a polynomial quasiregular mapping $g\colon \mathbb{R}^n \to N$, it suffices by the ideas of the previous section to show that the universal cover $\widetilde{N}$ of N is n-Ahlfors regular and n-Loewner when condition (1) of Theorem 1 holds.

The Ahlfors n-regularity follows from the facts that the universal cover is both locally and globally Ahlfors n-regular. The universal cover is locally isometric to the manifold N and thus locally Ahlfors n-regular. In the large scale the universal cover $\widetilde{N}$ resembles the fundamental group $\pi_1(N)$ of N – there exists a coarse quasi-isometry $\pi_1(N) \to \widetilde{N}$. Now a straightforward volume counting argument shows that $\widetilde{N}$ is n-Ahlfors.

The n-Loewner property of a space was first defined by Heinonen and Koskela in [33]. It states, heuristically, that there are no relatively narrow necks in the space. For example, the Euclidean space $\mathbb{R}^n$ has the n-Loewner property, while the set

$$\{(x,y) \in \mathbb{R}^2 : |y| \geq x\} \tag{2.1}$$

is not 2-Loewner. Instead of defining and discussing the Loewner condition further, we note that by a result of Heinonen and Koskela [33, Corollary 5.13] since $\widetilde{N}$ is Ahlfors n-regular, it is n-Loewner if and only if $\widetilde{N}$ supports a weak $(1,n)$-Poincaré inequality. By a (p,q)-Poincaré inequality we mean that there exists $C > 0$ and $\lambda \geq 1$ for which

$$\left(\fint_{B(x,r)} |f - f_{B(x,r)}|^p \right)^{\frac{1}{p}} \leq Cr \left(\fint_{B(x,\lambda r)} |\nabla f|^q \right)^{\frac{1}{q}} \tag{2.2}$$

for all $x \in \widetilde{N}$, $r > 0$ and $f \in C^\infty(\widetilde{N})$. Here we denote by $f_{B(x,r)}$ the average

$$\fint_{B(x,r)} f(y)\,\mathrm{d}y.$$

To give a few examples, as in the case of the Loewner condition, the Euclidean space $\mathbb{R}^n$ satisfies the $(1,n)$-Poincaré inequality, while the set given in (2.1) does not. A plethora of examples of non-manifold spaces satisfying the Poincaré inequality can be found from [49].

The $(1,n)$-Poincaré inequality follows immediately from a $(1,1)$-Poincaré inequality with a straightforward application of Hölder's inequality. As with the Ahlfors n-regularity, this $(1,1)$-Poincaré inequality in turn follows by showing that the space satisfies the inequality both locally and in large scale, see [10, Theoreme 7.2.(3)]. The local inequality is, again, result from the fact that the covering map $\widetilde{N} \to N$ is a local isomorphism, see e.g. [40, Lemma 8]. The large scale inequality, on the other hand, follows from the fact that a finitely generated group with polynomial growth rate satisfies a type of discrete $(1,1)$-Poincaré inequality.

Poincaré inequalities for polynomially growing finitely generated groups seem to have been considered as folklore; for the earliest result stating essentially the $(1,1)$-Poincaré inequality see [9, p. 308-309], for a complete proof see [62, Lemma 3.17] and for a $(2,2)$-Poincaré inequality see [41, Theorem 2.2].

3 BLD-mappings and covering maps

Recall that non-constant quasiregular- and BLD-mappings are always *branched covers*, i.e. continuous, open and discrete maps. The terminology rises from the fact that topologically these resemble covering maps; by the Cernavskii-Väisälä result [80] a branched cover between (generalized) n-manifolds is a local homeomorphism outside a closed set of topological dimension at most $n-2$. Also recall that a *covering map* is a surjective local homeomorphism $p\colon X \to Y$ for which there exists for any point $y \in Y$ a neighbourhood V such that the pre-image $f^{-1}V$ is a disjoint union $\bigcup_\alpha U_\alpha$ of domains with each of the restrictions $f|_{U_\alpha}$ a homeomorphism. As an example, note that the Alexander map defined in Section 1.1 is a covering map outside the labeled points A, B, C and D.

The focus of this section, and paper [B], is in asking how much do BLD-mappings resemble covering maps. For quasiregular mappings there is Zorich's theorem, see [83] or [77, Corollary 2.8, p.70], that states the following: *A locally homeomorphic quasiregular map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism.* From this we see that any (non-constant) quasiregular mapping from $\mathbb{R}^n$ to a quotient of $\mathbb{R}^n$ is a covering map when it is a local homeomorphism. Since in the Euclidean setting BLD-mappings form a subclass of quasiregular maps, the same result naturally holds for a locally homeomorphic BLD-mapping. The question giving rise to [B] is as follows: *Let $f\colon X \to Y$ be a locally homeomorphic BLD-mapping between metric spaces. When is f a covering map?*

3.1 Lifting paths

In the theory of covering spaces and covering maps, the *lifting of paths* is a basic tool. To fix the terminology, let $f\colon X \to Y$ be a continuous mapping and $\beta\colon [0,1] \to Y$ a path. For a point $x_0 \in f^{-1}\{\beta(0)\}$ and a subinterval $J \subset [0,1]$ with $0 \in J$ we say that a path $\alpha\colon J \to X$ is *a lift of β from the point x_0 under f*, if $\alpha(0) = x_0$ and $f \circ \beta|_J = \alpha$. The lift is *maximal* if the lift cannot be extended to a lift defined on a strictly larger interval $J' \supset J$, and a lift is called a *total lift* if $J = [0, 1]$.

For a local homeomorphism maximal lifts of paths always exist; a local lift always exists via the local inverse mapping and a straightforward Zorn's lemma argument shows that the existence of local lifts gives rise to maximal lifts, see e.g. [77, p. 33]. A less obvious result is that local lifts exist also for branched covers between locally compact path-metric spaces, see [15] for the general case and [75, 77] for branched covers between manifolds. Again the local lifts give rise to maximal lifts, so for BLD- and quasiregular mappings maximal lifts of paths always exist.

The existence of total lifts, on the other hand, is a much stronger condition. The rest of this section is mostly dedicated to questions about the existence of total lifts since it turns out that the covering property and the existence of total lifts of paths are connected. The following fundamental lemma is folklore, see e.g. [12, Proposition 6, p.383] for a proof in the setting of smooth surfaces.

Lemma 3. *Let $f\colon X \to Y$ be a local homeomorphism between path-connected topological Hausdorff spaces that are semilocally simply connected. Then f is a covering map if and only if for any path $\beta\colon [0, 1] \to Y$ there exists for each $x \in f^{-1}\{\beta(0)\}$ a path $\alpha\colon [0, 1] \to X$ with $f \circ \alpha = \beta$.*

Thus we see that for local homeomorphisms, the existence of *total* lifts is equivalent to the mapping being a covering map. We can rarely hope for a BLD-mapping to be a covering map, but we may ask whether total lifts of all paths exist – when they do it will be natural to think BLD-mappings as generalizations of covering maps, since in such a setting the locally homeomorphic BLD-mappings are covering maps.

We note first that for *rectifiable* paths total lifts exist under BLD-mappings. Indeed, for an L-BLD-mapping $f\colon M \to N$ and any rectifiable path $\beta\colon [0, 1] \to N$ we may take the maximal lift $\alpha\colon [0, t_0) \to M$ of β. By the (BLD) inequality this lift will be contained in the compact closed ball $\overline{B}_M(\alpha(0), L\ell(\beta))$, and so there exists a continuation $\alpha'\colon [0, t_0] \to M$ of α which is still a lift of β. For $t_0 < 1$ the existence of local lifts implies that the lift α' may be extended even further, which contradicts the maximality of α. Thus total lifts of rectifiable paths exist under BLD-mappings. Note that the lifts of rectifiable paths are always rectifiable under BLD-mappings.

In showing the existence of total lifts of rectifiable paths under BLD-mappings we only needed the rectifiability to guarantee that the maximal lift of a given path is contained in a compact set. This turns out to be a crucial property, and we give the following definition.

Definition. Let $f\colon X \to Y$ be a continuous mapping between topological spaces. We say that $y_0 \in Y$ is *an asymptotic value of* f if there exists a path $\alpha\colon [0,1) \to X$ such that $\lim_{t\to 1} f(\alpha(t)) = y_0$, but $\alpha[0,1)$ is not contained in any compact subset of X.

With this definition we see that the reason why rectifiable paths have total lifts under BLD-mappings is that the lift of any segment of a rectifiable path under a BLD-mapping cannot generate a path giving rise to an asymptotic value since rectifiability is preserved under BLD-mappings. In general the lack of asymptotic values gives rise to the existence of total lifts for BLD-mappings; for a BLD-mapping $f\colon M \to N$ between complete Riemannian n-manifolds all maximal lifts of paths are total if f has no asymptotic values.

Indeed, in [B] one of the main efforts was to study when BLD-mappings have no asymptotic values, see Corollary 1.5 of [B]. Note that polynomial quasiregular mappings $f\colon \mathbb{R}^n \to N$ have no asymptotic values if N is an oriented and complete Riemannian manifold that is both Ahlfors n-regular and n-Loewner by a result of Onninen and Rajala, [59, 12.1.(e)]. For BLD-mappings the non-existence of asymptotic values is not restricted to this setting.

In general quasiregular mappings may have abundantly many asymptotic values, for an extremal example see [13], where Drasin constructs a non-constant quasiregular map $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ for which every point in the range of f is an asymptotic value of f.

3.2 Multiplicity bounds for BLD-mappings

In [B] the main results are concerned in the existence of asymptotic values of BLD-mappings, for terminology see [B].

Theorem 4 ([B, Corollary 1.5]). *Let M and N be n-manifolds of bounded geometry and let $f\colon M \to N$ be a BLD-mapping. Then f has no asymptotic values.*

The proof of Theorem 4 relies on two facts. First that BLD-mappings are *Lipschitz quotient* mappings and second on certain uniform multiplicity bound of BLD-maps. The first fact is straightforward to prove in our setting, but for the multiplicity bound we need to have some basic results concerning the *branch set* of a BLD-mapping. Recall that for a branched cover $f\colon X \to Y$ the branch set B_f is the set of points at which f is not a local homeomorphism. For our needs an essential property of the branch set is that points outside the image of the branch set have the same (maximal) amount of pre-images, at least locally, and that the branch set of a BLD-mapping has measure zero. For this to hold we assume throughout the section that $f\colon M \to N$ is an L-BLD-mapping between two Ahlfors regular and complete Riemannian n-manifolds.

We note first the geometric fact that an L-BLD-mapping $f\colon M \to N$ between complete and locally compact path-metric spaces is an *L-LQ-mapping*, i.e. for all $x \in M$ and $r > 0$

$$B_Y(f(x), L^{-1}r) \subset fB_X(x,r) \subset B_Y(f(x), Lr).$$

The connection of the classes of BLD- and LQ-mappings is the focus of both Section 4 and paper [C].

Another important fact is the following pointwise equidistribution result.

Lemma 5 (C, Lemma 4.4.)**.** *Let $f\colon M \to N$ be a locally homeomorphic L-BLD-mapping between complete Riemannian n-manifolds. For any $x, y \in N \setminus fB_f$ there exists a bijection $\phi\colon f^{-1}\{x\} \to f^{-1}\{y\}$ such that $d(\phi(a), a) \le Ld(x,y)$ for all $a \in f^{-1}\{x\}$.*

The proof of Lemma 5 relies on two notions. Firstly, in the setting of manifolds there exists the notion of *maximal sequence of path-liftings*, see [77, p. 32] and secondly, the observation that the maximal lifts of a rectifiable path connecting x and y are total.

Lemma 5 gives rise to a multiplicity bound that was first observed by Martio and Väisälä in [53, Theorem 4.12], and later generalized in the setting of generalized manifolds of type A by Onninen and Rajala in [59, Theorem 6.8]. The form of the multiplicity bound we use is as follows, see also [B, Lemma 2.1].

Theorem 6. *Let $f\colon M \to N$ be a BLD-mapping between two generalized n-manifolds of type A equipped with a complete path-metric. Then for all $r_0 > 0$ there exists a number $N_0 \in \mathbb{N}$ for which*

$$\sup_{y \in B(x,r_0)} \# \left(B(x,r_0) \cap f^{-1}\{f(y)\} \right) \le N_0 \tag{N}$$

for all $x \in X$.

Theorem 6 follows from Lemma 5 when the branch set has zero measure. Indeed, if the branch has zero measure we can use basic covering theorems to cover almost all of the set $fB(x, r_0)$ by small balls B_j whose pre-images are L-bilipschitz equivalent to B_j. By the pointwise equidistribution lemma the amount of these pre-images is roughly the same. By the LQ-condition the volumes of $B(x, r_0)$ and $f(B(x, r_0))$ are the same up to a constant. Now a volume counting argument yields (N). For a descriptive picture see Figure 5.

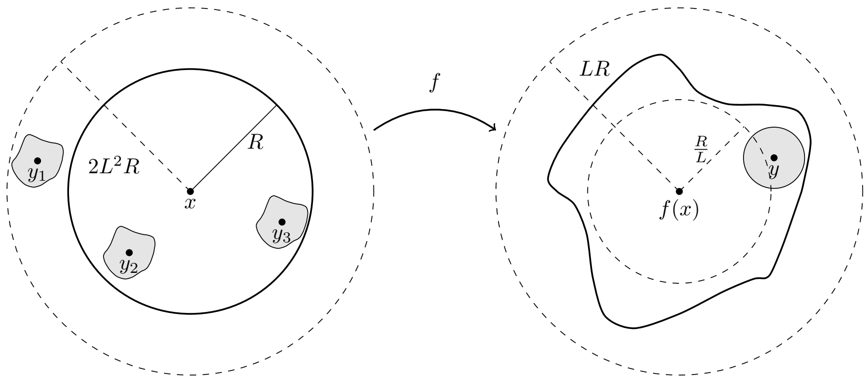

Figure 5: Picture concerning the multiplicity bound of Theorem 6.

3.3 Asymptotic values and complete spreads

Theorem 6 combined with the LQ-condition now forbids asymptotic values of BLD-mappings in our setting. Indeed, suppose $y_0 \in Y$ is an asymptotic value for an L-BLD-mapping $f\colon X \to Y$ and denote by $\gamma\colon [0,\infty) \to X$ the path corresponding to this asymptotic value. We take a unit ball B_t centered at $\gamma(t)$ and for any $k \in \mathbb{N}$ we fix k disjoint balls $B_t^1, \ldots, B_t^k \subset B_t$ with a joint radius $r_k = (2k+1)^{-1}$. Since f is L-LQ, the images of these disjoint balls B_t^j will contain balls of radius r_k/L. On the other hand since they lay on the path γ corresponding to the asymptotic value y_0, for large enough t the images of these balls all overlap at y_0. Thus

$$\#(B_t \cap f^{-1}\{y_0\}) \geq k,$$

which gives rise to a contradiction with our multiplicity bound.

A similar proof gives rise to a bound for the diameter of the components of pre-images of small balls under f. A mapping satisfying the conclusion of of the following Theorem 7 is called a *complete spread*, see e.g. [1, 16, 56]. For terminology see, again, [B].

Theorem 7 ([B, Theorem 1.5.]). *Let M and N be n-manifolds of bounded geometry and let $f\colon M \to N$ be an L-BLD-mapping. Then there exists a radius $R_{inj} > 0$ and constants $D > 0$ and $k \geq 1$ depending only on M, N and L such that for each $y \in N$ and any $0 < r \leq R_{inj}$ the pre-image $f^{-1}B_N(y,r)$ consists of pair-wise disjoint domains U with* $\operatorname{diam}(U) \leq Dr$ *for which the restriction $f|_U\colon U \to B_N(y,r)$ is a surjective k-to-one BLD-mapping.*

4 BLD-mappings in metric spaces

In this final section we turn to the theory BLD-mappings in metric spaces. Even though some of the ideas of in this section work in general metric spaces, we follow the spirit of paper [C] and throughout the rest of the section, unless otherwise specified, assume X and Y to be locally compact and complete path-metric spaces. In Section 3 we noted that between path-metric and complete generalized manifolds of type A locally homeomorphic BLD-mappings are covering maps and can thus be seen as a generalization of covering maps. In this section we study classes of mappings $\mathcal{F}$ such that locally homeomorphic mappings in $\mathcal{F}$ are locally L-bilipschitz. Recall that a mapping $f\colon X \to Y$ is *L-bilipschitz*, if for all x and y

$$L^{-1}d(x,y) \leq d(f(x),f(y)) \leq Ld(x,y). \tag{BL}$$

Similarly as in Section 3 such classes of mappings can then be seen as generalizations of locally L-bilipschitz mappings. Even though not immediately apparent from the (BLD) condition, locally homeomorphic L-BLD-mappings are locally L-bilipschitz. This fact comes clear after our main result which is a characterization theorem of BLD-mappings via these classes of mappings.

To find classes of mappings for which locally homeomorphic mappings belonging to these classes are locally L-bilipschitz we study how locally L-bilipschitz mappings act in a small ball.

4.1 Radial mappings

To quantify the behaviour of a mapping $f\colon X \to Y$ in a ball we set

$$\operatorname{Lip}(x,f,r) := \sup_{y \in B(x,r)} \frac{d(f(x),f(y))}{d(x,y)}$$

and

$$\operatorname{lip}(x,f,r) := \inf_{y \in B(x,r)} \frac{d(f(x),f(y))}{d(x,y)}.$$

These numbers measure the pointwise behaviour of the mapping and resemble a difference quotient. In fact, taking the limit $r \to 0$ does give rise to concepts in Lipschitz analysis like the metric differential. We, however, will be satisfied with the local bounds. For a locally L-bilipschitz map f we have

$$L^{-1} \leq \operatorname{lip}(x,f,r_x) \leq \operatorname{Lip}(x,f,r_x) \leq L \tag{4.1}$$

for all $x \in X$ and $r_x > 0$ small enough. On the other hand, a local homeomorphism satisfying (4.1) for all $x \in X$ and $r_x > 0$ small enough will be a local L-bilipschitz mapping; as with L-BLD-mappings, this is not immediately obvious, but follows e.g. from Theorem 10, which is the main result in the paper [C]. This means that mappings satisfying (4.1) form a class of mappings generalizing locally L-bilipschitz mappings:

Definition 8. A mapping $f\colon X \to Y$ is an *L-radial mapping* if it satisfies (4.1) for all $x \in X$ and $r_x > 0$ small enough.

Radial mappings need not be local homeomorphisms. For example the winding map w in Section 1.1 and the *the folding map*

$$F\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad (x,y) \mapsto (|x|, y)$$

are L-radial but not local homeomorphisms. We note that an equivalent way of defining L-radial mappings is to require that for all $x \in X$ there exists a radius $r_x > 0$ such that

$$L^{-1}d(x,y) \leq d(f(x), f(y)) \leq Ld(x,y) \tag{R}$$

for all $y \in B(x, r_x)$. Note the strong resemblance of (R) to the bilipschitz condition (BL).

In Section 1.3 we studied the distortion of an ellipsoid both from an intrinsic and an extrinsic point of view. The intrinsic approach studied the ratio of the minor and major axes of the ellipsoid, while the extrinsic point of view focused just on the volume of the ellipsoid via the Jacobian. Here the definition of L-radial mappings represents the intrinsic approach as we study the pointwise behaviour of the mapping whilst ignoring the shape of the image; e.g. the 1-radial mapping $(x,y) \mapsto (|x|, |y|)$ maps the unit ball $B(0,1)$ to a quarter-ball which does not metrically resemble a ball in our sense.

4.2 Lipschitz quotient mappings

Next we turn to the extrinsic point of view; we focus on the image of the ball while ignoring the pointwise behaviour. In a general metric space the concept of volume is not a natural approach, but in Section 1.3 we noted that for an ellipsoid the volume was bounded by the volumes of balls containing- and being contained in the ellipsoid; recall Figure 3. Thus instead of studying the volume of the image of a ball we can ask the radii of the largest and smallest ball containing and being contained in the image of a ball. For an L-Lipschitz mapping the

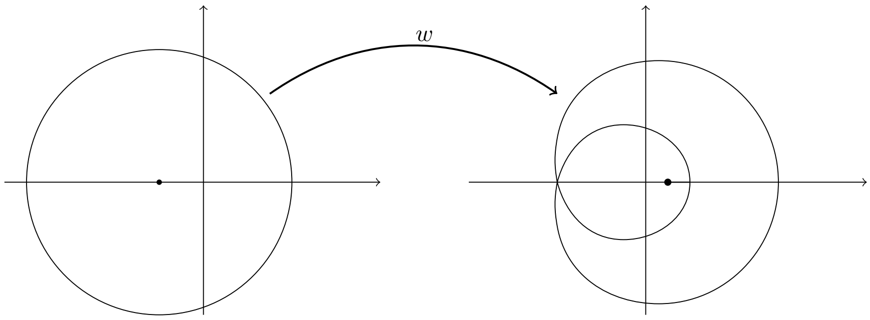

Figure 6: Image of $B((-1,0),3)$ under the 2-to-1 winding map w.

image of a ball $B(x,r)$ is contained in the ball $B(f(x),Lr)$. For a locally L-bilipschitz mapping a dual condition called the *co-Lipschitz property* also holds; the image of any ball $B(x,r)$ contains the ball $B(f(x),L^{-1}r)$. Combining these two we have

$$B_Y(f(x),L^{-1}r) \subset fB_X(x,r) \subset B_Y(f(x),Lr) \tag{LQ}$$

for all $x \in X$ and $r > 0$ *small enough* – after a moment's thought, however, we see that the condition (LQ) holds for all $r > 0$; indeed, take a point $z \in X$ and assume that the (LQ)-condition holds at z for some bounded collection $\{r_s\}$ of radii. A complete and locally compact path-metric space is a proper geodesic space, so the (LQ)-condition holds at z also for the supremum r_0 of the radii $\{r_s\}$. Furthermore in a proper metric space the set $\partial B_X(z,r_0)$ is compact and can thus be covered with a finite collection of balls in which the (LQ)-condition holds. From this we can conclude that the collection of radii for which the (LQ)-condition holds is a non-empty open and closed subset of $(0,\infty)$ and thus all of $(0,\infty)$.

On the other hand we see that a local homeomorphism satisfying (LQ) is a locally L-bilipschitz mapping, since the condition (LQ) is required for all radii. Motivated by this we give the following definition.

Definition 9. A mapping $f\colon X \to Y$ is *L-LQ (Lipschitz Quotient)* if it satisfies the (LQ)-condition.

We again point out that syntactically the condition (LQ) resembles the (BL) condition. See also Figure 4 for the canonical picture describing LQ-mappings.

As mentioned, radial- and LQ-mappings correspond to an intrinsic and an extrinsic approaches to generalizing locally bilipschitz mappings. To underline the differences of the approaches, let us look at the winding map $w\colon \mathbb{R}^2 \to \mathbb{R}^2$ and the image of the ball $B((-1,0),3)$ under w, see Figure 6. The winding map is both 2-radial and 2-LQ. The difference in the definitions is in the fact that for radial mappings we are interested in the behaviour of the image of the boundary of small balls while with LQ-mappings we study the boundary of the image of a small ball. See Figure 7.

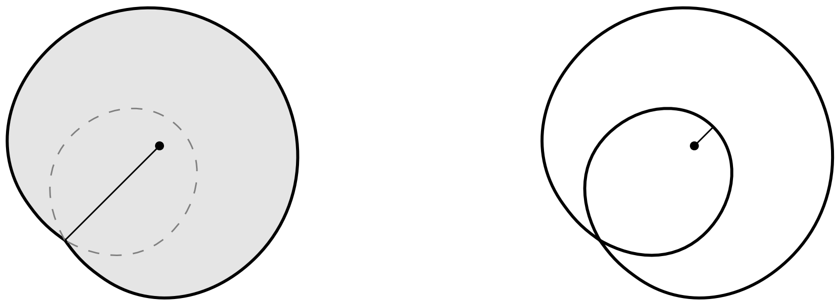

Figure 7: Boundary of the image and image of the boundary.

4.3 A characterization result

The main result of [C] ties these classes of mapping together.

Theorem 10 ([C, Theorem 1.1.]). *Let $f\colon X \to Y$ be a continuous mapping between two complete locally compact path-metric spaces and $L \geq 1$. Then the following are equivalent:*

(i) f is an L-BLD-mapping,

(ii) f is a discrete L-LQ-mapping, and

(iii) f is an open L-radial mapping.

As mentioned, in the Euclidean setting this result is due to [53]. The implication $(i) \Rightarrow (ii)$ is observed in the setting of generalized manifolds of type A in [35], and more recently the implication $(i) \Rightarrow (iii)$ is noted in a setting similar to [35] under additional assumptions on spaces X and Y by Guo and Williams

[28]. Furthermore the implication $(iii) \Rightarrow (ii)$ is implicitly due to Lytchak in a purely metric setting without notions of branched covers, see [48, Section 3.1. and Proposition 4.3].

Via the characterization Theorem 10 we finally see that locally homeomorphic L-BLD- and L-radial mappings are locally L-bilipschitz, since L-LQ mappings are. Besides being a natural non-homeomorphic generalization of bilipschitz mappings, the classes of BLD-, radial- and LQ-mappings can be heuristically interpreted as *metric versions of topological concepts.* In the same sense as the Lipschitz condition can be thought as a metrically quantified version of continuity we may do the following comparisons:

Topological	**Metric**
Continuous	Lipschitz
Open	co-Lipschitz
Discrete	Radial
Branched cover	BLD

4.4 Limit results of BLD-mappings

Already in [53] Martio and Väisälä prove that in the setting of Euclidean domains a locally uniform limit of L-BLD mappings is L-BLD. In [30] Heinonen and Keith show that in the setting of generalized manifolds of type A such a limit is L'-BLD for some L' depending only on the data. The proof by Martio and Väisälä in [53] relies to the characterization of BLD-mappings as quasiregular mappings. The proof by Heinonen and Keith in [30], on the other hand, relies on the fact proven in [35, Theorem 4.5] that a mapping between quasiconvex generalized manifolds is BLD if and only if it is locally regular in the sense of David and Semmes, see [11, Definition 12.1]. That equivalence is of quantitative nature but not sharp with respect to the parameter L – this gives rise to the fact that their limit theorem is also not sharp.

The characterization of L-BLD-mappings as discrete L-LQ-mappings gives rise to sharp BLD-limit results, since the locally uniform limit of L-LQ mappings is L-LQ in a quite general context; indeed, for e.g. ultralimits, see [47] and for pointed mapping package limits see [C, Theorem 1.4]. The idea of the proof in these two very general cases is essentially the same as with (locally) uniform limits of L-LQ-mappings f_j between Euclidean domains. From the limit results of L-LQ mappings we see that the locally uniform limit of L-BLD-mappings is L-BLD whenever the limiting map is discrete.

Theorem 11 ([C, Corollary 1.3]). *Let X and Y be locally compact complete path-metric spaces and suppose (f_j) is a sequence of L-BLD-mappings $X \to Y$ converging locally uniformly to a continuous discrete mapping $f\colon X \to Y$. Then f is L-BLD.*

The discreteness of the limit map is a non-trivial property. In the Euclidean setting Martio and Väisälä in [53] show the discreteness by first proving that the limiting map is a non-constant quasiregular mapping and then applying Reshetnyak's Theorem – a quasiregular mapping is either a branched cover or a constant. The method of Heinonen and Keith in [30], on the other hand relied on the class of regular mappings that contain, in their definition, strong uniform multiplicity bounds which give discreteness. This idea can be imitated to prove discreteness results – indeed in Section 3.2 we showed that BLD-mappings between path-metric and complete generalized manifolds of type A satisfy the uniform multiplicity bound (N). This gives discreteness of limiting maps in limits of mapping packages, for fixed spaces we can state the following.

Theorem 12 ([C, Corollary 1.5]). *Let M and N be generalized n-manifolds of type A and suppose (f_j) is a sequence of L-BLD-mappings $M \to N$ converging locally uniformly to a continuous mapping $f\colon M \to N$. Then f is L-BLD.*

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