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On infinite-dimensional Banach spaces and weak forms of the axiom of choice

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We study theorems from Functional Analysis with regard to their relationship with various weak choice principles and prove several results about them: "Every infinite-dimensional Banach space has a well-orderable Hamel basis" is equivalent to AC; "$\mathbb{R}$ can be well-ordered" implies "no infinite-dimensional Banach space has a Hamel basis of cardinality $< 2^{\aleph_0}$", thus the latter statement is true in every Fraenkel-Mostowski model of ZFA; "No infinite-dimensional Banach space has a Hamel basis of cardinality $< 2^{\aleph_0}$" is not provable in ZF; "No infinite-dimensional Banach space has a well-orderable Hamel basis of cardinality $< 2^{\aleph_0}$" is provable in ZF; $\mathsf{AC}^{\aleph_0}_{\mathsf{fin}}$ (the Axiom of Choice for denumerable families of non-empty finite sets) is equivalent to "no infinite-dimensional Banach space has a Hamel basis which can be written as a denumerable union of finite sets"; Mazur's Lemma ("If X is an infinite-dimensional Banach space, Y is a finite-dimensional vector subspace of X, and $\varepsilon > 0$, then there is a unit vector $x \in X$ such that $||y|| \leq (1+\varepsilon)||y + \alpha x||$ for all $y \in Y$ and all scalars α") is provable in ZF; "A real normed vector space X is finite-dimensional if and only if its closed unit ball $B_X = \{x \in X : ||x|| \leq 1\}$ is compact" is provable in ZF; DC (Principle of Dependent Choices) + "$\mathbb{R}$ can be well-ordered" does not imply the Hahn-Banach Theorem (HB) in ZF; HB and "no infinite-dimensional Banach space has a Hamel basis of cardinality $< 2^{\aleph_0}$" are independent from each other in ZF; "No infinite-dimensional Banach space can be written as a denumerable union of finite-dimensional subspaces" lies in strength between $\mathsf{AC}^{\aleph_0}$ (the Axiom of Countable Choice) and $\mathsf{AC}^{\aleph_0}_{\mathsf{fin}}$; DC implies "No infinite-dimensional Banach space can be written as a denumerable union of closed proper subspaces" which in turn implies $\mathsf{AC}^{\aleph_0}$; "Every infinite-dimensional Banach space has a denumerable linearly independent subset" is a theorem of $\mathsf{ZF} + \mathsf{AC}^{\aleph_0}$, but not a theorem of ZF; and "Every infinite-dimensional Banach space has a linearly independent subset of cardinality $\geq 2^{\aleph_0}$" implies "every Dedekind-finite set is finite".

1 Notation and terminology

We start with the definitions of notions that will be used in the sequel.

Definition 1.1 Let $(V, +, \cdot)$ be a vector space over a field F.

1. If $X \subseteq V$, then $\langle X \rangle$ denotes the *linear span* of X, i.e., the subspace of V which consists of all finite linear combinations of elements of X.
2. A set $B \subseteq V$ is called a *Hamel basis* (or simply a *basis*) for V if B is linearly independent and $V = \langle B \rangle$. (If B is a Hamel basis for V, then every vector $v \in V$ can be expressed uniquely as a finite linear combination of elements of B.)

It is a standard result, taught in every undergraduate Linear Algebra course, that every finitely generated vector space (i.e., every vector space which is spanned by a finite set of vectors) has a basis, and that all bases of a finitely generated vector space are equipotent: the number of elements of any basis of a finitely generated vector space V is called the *dimension* of V and it is denoted by $\dim(V)$.

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Definition 1.2 A vector space $(V, +, \cdot)$ over a field F is called *finite-dimensional* if V is finitely generated. Otherwise, V is called *infinite-dimensional*.

The result that every finite-dimensional vector space (over any field) has a basis is certainly a theorem of ZF, that is, in order to construct a basis for a given finite-dimensional vector space, one does not need to invoke any form of choice. However, the Axiom of Choice AC is indispensable in the theory of infinite-dimensional spaces. In particular, strange phenomena such as the existence of a (infinite-dimensional) vector space with no Hamel basis or a vector space which has two bases of different cardinalities are relatively consistent with set theory without AC (cf., e.g., [8, Theorem 10.11 and Exercise 5, p. 149]). Furthermore, it is a renowned result of Blass [4] that, in ZF, AC is equivalent to "for every field F, every vector space over F has a basis".

Definition 1.3 Let (X, d) be a metric space.

1. X is *complete* if every Cauchy sequence of elements of X converges in X.
2. X is *Baire* if it cannot be written as a countable union of nowhere dense sets (i.e., sets whose closure has empty interior).
3. X is *compact* if every open cover of X has a finite subcover.
4. X is *separable* if it has a countable dense subset.

Definition 1.4 1. A *Banach space* is a complete normed vector space over $\mathbb{R}$, i.e., a real vector space with a norm which is complete with respect to the metric induced by the norm.

2. A vector space X over $\mathbb{R}$ with an inner product is a *Hilbert space* if X is complete with respect to the norm induced by the inner product.

3. Let I be any set and let $H = \ell^2(I)$ denote the set of all functions $x : I \to \mathbb{R}$ such that $x(i) = 0$ for all but a countable number of i and $\sum_{i \in I}(x(i))^2 < \infty$ (due to the definition of x this is the usual convergence of series), which is equipped with pointwise operations of addition and scalar multiplication (with scalars in $\mathbb{R}$). For x and y in H define $\langle x, y\rangle = \sum_{i \in I} x(i)y(i)$. It is well-known (cf. [9,16]) that H is a Hilbert space (i.e., $\langle\ ,\ \rangle$ is an inner product and H is a Banach space with the induced norm $||x|| = \sqrt{\langle x, x\rangle}$).

Definition 1.5 1. A set X is called *finite* if there exists a natural number n and a bijection $f : n \to X$. Otherwise, X is called *infinite*.

2. A set X is called *Dedekind-finite* if there is no one-to-one mapping $f : \omega \to X$. Otherwise, X is called *Dedekind-infinite*.

Equivalently, X is Dedekind-finite if there is no one-to-one mapping from X into a proper subset of X.

3. An infinite set X is called *amorphous* if X cannot be expressed as a disjoint union of two infinite subsets.

Definition 1.6 1. AC is the *Axiom of Choice*, i.e., for every set X of non-empty sets there is a function f with domain X such that for every $x \in X$, $f(x) \in X$. Such a function f is called a *choice function* for X.

2. MC is the *Axiom of Multiple Choice*, i.e., for every set X of non-empty sets there is a function f with domain X such that for every $x \in X$, $f(x)$ is a non-empty finite subset of x. Such a function f is called a *multiple choice function* for X.

3. $\mathsf{MC}^{\aleph_0}$ is MC restricted to denumerable families of non-empty sets. It is known (cf. [6]) that $\mathsf{MC}^{\aleph_0}$ is equivalent to "every denumerable family $\mathcal{A}$ of non-empty sets has a partial multiple choice function (i.e., there is an infinite subfamily $\mathcal{B}$ of $\mathcal{A}$ with a multiple choice function)"

4. $\mathsf{AC}^{\aleph_0}$ is the *Axiom of Countable Choice*, i.e., AC restricted to denumerable (= countably infinite) families of non-empty sets. It is known (cf. [6]) that $\mathsf{AC}^{\aleph_0}$ is equivalent to "every denumerable family $\mathcal{A}$ of non-empty sets has a partial choice function".

5. $\mathsf{AC}^{\aleph_0}_{\mathsf{fin}}$ is AC restricted to denumerable families of non-empty finite sets. It is known (cf. [6]) that $\mathsf{AC}^{\aleph_0}_{\mathsf{fin}}$ is equivalent to "every denumerable family of non-empty finite sets has a partial choice function".

6. $\mathsf{PKW}^{\aleph_0}_{\mathsf{fin},\geq 2}$ is "every denumerable family $\mathcal{A}$ of finite sets each with at least two elements has a partial Kinna-Wagner selection function", i.e., there is an infinite subfamily $\mathcal{B}$ of $\mathcal{A}$ and a function f with domain $\mathcal{B}$ such that for every $B \in \mathcal{B}$, $f(B)$ is a non-empty proper subset of B (f is a partial Kinna-Wagner selection function for $\mathcal{A}$—and a Kinna-Wagner selection function for $\mathcal{B}$).

7. Let n be an integer greater than or equal to 2. $\mathsf{PAC}^{\aleph_0}_{\leq n}$ is "every denumerable family of non-empty sets each with at most n elements has a partial choice function".

8. DF=F is "every Dedekind-finite set is finite". It is known (cf. [6]) that $\mathsf{AC}^{\aleph_0}_{\mathsf{fin}}$ is strictly weaker than DF=F in ZF.

9. $\mathsf{AC}_{\mathbb{R}}$ is AC restricted to families of non-empty sets of reals. $\mathsf{AC}_{\mathbb{R}}$ is equivalent to the statement "$\mathbb{R}$ can be well-ordered" (cf. [6]).

10. DC is the *Principle of Dependent Choices*: Let R be a binary relation on a non-empty set A such that $(\forall x \in A)(\exists y \in A)(x\ R\ y)$. Then there is a sequence $(x_n)_{n\in\omega}$ of elements of A such that $x_n\ R\ x_{n+1}$ for all $n \in \omega$.

11. Let κ be an infinite well-ordered cardinal number. W_κ is the statement: For every set X, either $|X| \le \kappa$ or $\kappa \le |X|$. It is known (cf. [6, 8]) that "$(\forall\kappa)\mathsf{W}_\kappa$" is equivalent to AC.

12. CH is the *Continuum Hypothesis*, i.e., $2^{\aleph_0} = \aleph_1$.

13. BPI is the *Boolean Prime Ideal Theorem*: Every non-trivial Boolean algebra has a prime ideal.

14. OP is the *Ordering Principle*: Every set can be linearly ordered.

15. BCT is the *Baire Category Theorem*: Every complete metric space is Baire. It is a beautiful result of Blair (cf. [3] or [5, Theorem 4.106, p. 105]) that BCT is, in ZF, equivalent to DC.

We note that BCT is a theorem of ZF, if we restrict to the class of separable complete metric spaces, since these spaces are second countable, i.e., they have a countable base for their topology (the proof of [14, Theorem 7.2, pp. 294–295] goes through, in ZF, with the obvious adaptations).

16. HB is the *Hahn-Banach Theorem*: Suppose M is a subspace of a real vector space X, $p : X \to \mathbb{R}$ is a sublinear functional (i.e., $p(x + y) \le p(x) + p(y)$ and $p(tx) = tp(x)$, if $x, y \in X$ and $t \ge 0$), and $f : M \to \mathbb{R}$ is a linear functional such that $f(x) \le p(x)$ for every $x \in M$. Then there exists a linear functional $F : X \to \mathbb{R}$ such that $F \restriction M = f$ and $-p(-x) \le F(x) \le p(x)$ for every $x \in X$.

We recall here that BPI implies HB (the implication is not reversible in ZF) and that DC does not imply HB in ZF (cf. [6]).

A well-known consequence of HB is the *Hahn-Banach Theorem for normed spaces*: Let X be a normed vector space, let Y a linear subspace of X, and let $f : Y \to \mathbb{R}$ a bounded linear functional, that is

$$||f||_{Y^*} = \sup_{x\in Y, ||x||=1} |f(x)| < \infty,$$

where Y^* is the dual space of Y consisting of all bounded linear functionals on Y equipped with the above norm. (We also recall here that for all $y \in Y$, $|f(y)| \le ||f||_{Y^*}||y||$.) Then there exists a bounded linear functional $\tilde{f} : X \to \mathbb{R}$ such that $\tilde{f} \restriction Y = f$ and $||\tilde{f}||_{X^*} = ||f||_{Y^*}$.

In particular, if X is a non-trivial normed vector space and $x_0 \in X\backslash\{0\}$, then there exists a bounded linear functional $\tilde{f} : X \to \mathbb{R}$ such that $||\tilde{f}||_{X^*} = 1$ and $\tilde{f}(x_0) = ||x_0||_X$.

17. Let $(B, \oplus, \odot, 0_B, 1_B)$ be a Boolean algebra. A *real valued measure* on B is a mapping $m : B \to [0, 1]$ such that $m(1_B) = 1$ and m is finitely additive, i.e., for every $x, y \in B$, if $x \odot y = 0_B$, then $m(x \oplus y) = m(x) + m(y)$. (It follows that $m(0_B) = 0$.)

We shall also use the following standard notation: As usual, ω denotes the set of natural numbers; ZF is Zermelo-Fraenkel set theory without AC; ZFC is ZF + AC; ZFA is ZF with the Axiom of Extensionality modified in order to allow the existence of atoms; if X is a set, then $[X]^{<\omega}$ denotes the set of finite subsets of X; and if I and J are sets and λ is an infinite well-ordered cardinal, then $\mathrm{Fn}(I, J, \lambda)$ denotes the set of all partial functions p from I into J with $|p| < \lambda$ (i.e., there is a one-to-one mapping $f : p \to \lambda$, but no one-to-one mapping $g : \lambda \to p$).

2 Introduction and aims

Many (or most) of the results in Banach space theory, and in mathematical analysis in general, are provable in ZF + DC set theory; among the (rather) few exceptions is the Hahn-Banach Theorem HB. For example, we recall that BCT (hence DC) is the main apparatus used to prove fundamental theorems of Functional Analysis like the *closed graph theorem*, which in turn implies the *uniform boundedness principle*. To the best of our knowledge, it is an *open problem* whether "closed graph theorem" implies DC or even "uniform boundedness principle" implies $\mathsf{AC}^{\aleph_0}$.

A part of the folklore in Functional Analysis is that *if X is an infinite-dimensional Banach space, then X has no denumerable Hamel basis*, cf., e.g., [16, Exercise 1, p. 53]. The standard proof in the literature of this very nice—and striking—result uses BCT. However, BCT is a weak form of choice, thus a proof which employs the latter form cannot be considered as a constructive one. It is therefore interesting and natural to ask whether one actually needs any form of choice in order to derive that no infinite-dimensional Banach space admits a denumerable Hamel basis. One of the aims of this paper is to establish that *the answer to the above open problem is that no choice is needed*, and consequently the above functional analytic result is a theorem of ZF set theory. In fact, we shall prove a stronger result, namely, in ZF, no infinite-dimensional Banach space has a well-orderable Hamel basis of cardinality $< 2^{\aleph_0}$.

We shall also consider the following closely related propositions, all provable from BCT:

(1) no infinite-dimensional Banach space has a Hamel basis which is written as a denumerable union of finite sets,
(2) no infinite-dimensional Banach space can be written as a denumerable union of finite-dimensional subspaces,
(3) no infinite-dimensional Banach space can be written as a denumerable union of closed proper subspaces.

We shall show that the situation with the above three propositions is *strikingly different*. In particular, we shall establish that none of them can be proved without using some form of choice. Regarding statement (1), we shall actually give its exact characterization in terms of weak choice principles. Specifically, it turns out that (1) is equivalent to $\mathsf{AC}_{\mathrm{fin}}^{\aleph_0}$ (AC restricted to denumerable families of non-empty finite sets). Regarding statements (2) and (3), we shall show that $\mathsf{DC} \rightarrow (3) \rightarrow \mathsf{AC}^{\aleph_0} \rightarrow (2) \rightarrow \mathsf{AC}_{\mathrm{fin}}^{\aleph_0}$.

In the realm of infinite-dimensional Banach spaces and in view of the fact that these spaces do not possess a denumerable Hamel basis, it is natural to consider the following generalizations:

(a) no infinite-dimensional Banach space has a Hamel basis of cardinality $< 2^{\aleph_0}$,
(b) no infinite-dimensional Banach space has a well-orderable Hamel basis of cardinality $< 2^{\aleph_0}$.

Clearly, (a) $\rightarrow$ (b), but we shall establish that (a) is not a theorem of ZF and also that (a) is provable in $\mathsf{ZF} + \mathsf{AC}_{\mathbb{R}}$ (hence, it is true in every Fraenkel-Mostowski permutation model of ZFA), whereas in contrast with the non-provability of (a) in ZF, we shall show that (b) *is* a theorem of ZF. For the proof of (a) (from $\mathsf{AC}_{\mathbb{R}}$) we shall present two proofs, one of them based on Mazur's Lemma ("Let X be an infinite-dimensional Banach space, let Y be a finite-dimensional vector subspace of X, and let $\varepsilon > 0$. Then there is a unit vector $x \in X$ such that $||y|| \leq (1 + \varepsilon)||y + \alpha x||$ for all $y \in Y$ and all scalars α"), which is the key for the proof of Banach's important theorem that every infinite-dimensional Banach space has a basic sequence (cf., e.g., [1, Lemma 1.42 and Theorem 1.43, p. 30]). The well-known proof of Mazur's Lemma is conducted in ZFC (cf., e.g., [1, Lemma 1.42] or [13, Lemma 1.a.6]) using the Hahn-Banach Theorem and the fact that the closed unit ball of a finite-dimensional normed vector space is compact. In this paper, we shall establish that both Mazur's Lemma and the statement "a normed real vector space X is finite-dimensional if and only if the closed unit ball $B_X = \{x \in X : ||x|| \leq 1\}$ of X is compact" are provable in ZF. We point out that all known proofs to us of the above second fact are given in ZFC.

In this paper, we shall also construct a symmetric model N which satisfies $\mathsf{DC} + \mathsf{AC}_{\mathbb{R}}$, hence N satisfies statement (a), but N fails to satisfy HB. We note that the status of the implication "$\mathsf{DC} + \mathsf{AC}_{\mathbb{R}} \rightarrow \mathsf{HB}$" is stated as *unknown* in [6], hence our result settles the problem.

In this paper, we shall also consider the following related propositions:

(c) every infinite-dimensional Banach space has a linearly orderable Hamel basis,
(d) every infinite-dimensional Banach space has a well-orderable Hamel basis,
(e) every infinite-dimensional Banach space has a linearly independent subset of cardinality $\geq 2^{\aleph_0}$,
(f) every infinite-dimensional Banach space has a denumerable linearly independent subset

and we shall prove that (c) $\rightarrow$ OP and (d) $\leftrightarrow$ AC. We shall also prove that (f), hence (e), are not provable in ZF, and even more we shall find their placement in the hierarchy of weak choice principles. We should like to point out that *it is unknown* whether "every infinite-dimensional Banach space has a Hamel basis" implies AC.

In view of the above discussion, the major goal of this paper is to investigate and shed some light to the rather unexplored topic of the existence of Hamel bases and their cardinality as well as the existence of (infinite) linearly independent sets and their cardinality in infinite-dimensional Banach spaces in terms of weak choice principles. We believe that the results in the paper can be of interest not only to people in the foundations of mathematics but also to a more general mathematical audience.

Finally, we should like to point out that the research in the current paper also aims to clarify that studying fundamental theorems of Functional Analysis from the perspective of choice principles and their use or non-use in the related proofs, provides deeper insight, understanding, and appreciation for the importance of these theorems.

In the paper, most vector spaces shall be assumed to be *vector spaces over* $\mathbb{R}$. In any other case, we shall explicitly state which field is used.

3 Results in ZF only

In this section, we shall prove that both of the following propositions,

1. a normed real vector space X is finite-dimensional if and only if its closed unit ball B_X is compact, and
2. Mazur's Lemma,

are provable in ZF. We begin with some lemmas, whose proofs rely on some known ZF-facts, which we provide below.

(1) If $(X, ||\cdot||_X)$ and $(Y, ||\cdot||_Y)$ are two normed vector spaces, and if $f : X \to Y$ is a linear mapping which is a topological homeomorphism, then (in ZF) the mapping $||\cdot|| : X \to \mathbb{R}$ defined by $||x|| = ||f(x)||_Y$ for all $x \in X$, is a norm on X which is equivalent to $||\cdot||_X$, i.e., there exist real numbers C, C' such that

$$||x|| \le C||x||_X \ \text{ and } \ ||x||_X \le C'||x|| \ \text{ for all } \ x \in X,$$

or equivalently,

$$||f(x)||_Y \le C||x||_X \ \text{ and } \ ||x||_X \le C'||f(x)||_Y \ \text{ for all } \ x \in X.$$

The latter fact can be easily established using the continuity of f and f^{-1}. (Since f is continuous at 0, there is a $\delta > 0$ such that for all $x \in X$ with $||x||_X < \delta$ we have $||f(x)||_Y < 1$. Then for each $x \in B_X$ we have $||(\delta/2)x||_X = (\delta/2)||x||_X \le \delta/2 < \delta$ hence $||f((\delta/2)x)||_Y < 1$, so $||f(x)||_Y < 2/\delta$. It follows that for all $x \in X$, $||f(x/||x||_X)||_Y < 2/\delta$, hence $\|x\| = ||f(x)||_Y < (2/\delta)||x||_X$. Let $C := 2/\delta$. Using the continuity of f^{-1}, a similar argument can be used in order to obtain a $C' \in \mathbb{R}$ which witnesses the second inequality.)

It readily follows that given a subset A of X,

(i) A is complete in (the metric space) X if and only if $f[A]$ is complete in (the metric space) Y,
(ii) A is bounded in (the metric space) X if and only if $f[A]$ is bounded in (the metric space) Y,
(iii) A is closed in (the metric space) X if and only if $f[A]$ is closed in (the metric space) Y,
(iv) A is compact in (the metric space) X if and only if $f[A]$ is compact in (the metric space) Y.

(2) If n is a positive integer and X is a n-dimensional normed real vector space, then every algebraic isomorphism of X onto $\mathbb{R}^n$ (the standard n-dimensional normed real vector space) is a topological homeomorphism (cf. [16, Theorem 1.21, p. 16]). In particular, if $\{x_1, x_2, \ldots, x_n\}$ is a Hamel basis for X, then the algebraic isomorphism $f : X \to \mathbb{R}^n$ defined by $f(a_1x_1 + \cdots + a_nx_n) = (a_1, \ldots, a_n)$ is a topological homeomorphism.

(3) From facts (1) and (2), it follows that, in ZF, given a n-dimensional normed real vector space X (where n is some positive integer), any two norms on X are equivalent (if $||\cdot||_1$ and $||\cdot||_2$ are two norms on X and if $f : X \to \mathbb{R}^n$ is the above isomorphism which is a topological homeomorphism, then the norm $||\cdot||$ on X defined in fact (1) above (with $Y = \mathbb{R}^n$) is equivalent to both $||\cdot||_1$ and $||\cdot||_2$).

In view of facts (1) to (3), we easily obtain the results of the subsequent Lemma 3.1), Corollary 3.2) and Lemma 3.4.

Lemma 3.1 (ZF) *Let X be a (real) normed space and let Y be a finite-dimensional subspace of X. Then Y is complete. In particular, every finite-dimensional normed space is complete, and thus a Banach space.*

Corollary 3.2 (ZF) *Let $(X, ||\cdot||)$ be a finite-dimensional normed vector space and let Y be a non-empty subset of X. Then Y is compact if and only if it is closed and bounded.*

Proof. Assume that $\dim(X) = n$ for some positive integer n. Let $f : X \to \mathbb{R}^n$ be the algebraic isomorphism defined in fact (2) above (with respect to a prescribed Hamel basis $\{x_1, x_2, \dots, x_n\}$ for X), which is a topological homeomorphism. We also recall here the following well-known ZF-fact (cf. [6]) that, in ZF, a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

By fact (1), we have Y is compact in X if and only if $f[Y]$ is compact in $\mathbb{R}^n$ if and only if $f[Y]$ is closed and bounded in $\mathbb{R}^n$ if and only if Y is closed and bounded in X. □

Lemma 3.3 (Riesz; ZF) *If $(X, ||\cdot||)$ is a normed vector space, and if Y and Z are subspaces of X such that Y is a closed proper subspace of Z, then for all $\vartheta \in (0, 1)$ there is a $z \in Z$ such that $||z|| = 1$ and $d(z, Y) = \inf\{||z - y|| : y \in Y\} \geq \vartheta$, where d is the metric on X induced by the norm $||\cdot||$.*

Proof. Cf. [9, Lemma 2.5-4, p. 78]. □

We show next that, in ZF, a finite-dimensional subspace of a normed vector space X is closed in X. In ZFC, and in view of Lemma 3.1, the latter result is straightforward. A ZF-argument for the above result can also be found in [16, Theorem 1.21]. However, we prefer to provide our own proof in order to exploit further ideas and to give more information to the reader. In particular, we shall first prove, within ZF, the result of the subsequent lemma, which appears as [9, Lemma 2.4-1, p. 72], but its proof there is carried out in ZFC.

Lemma 3.4 (ZF) *Let $(X, ||\cdot||)$ be a normed space and let $x_1, \dots, x_n$ be linearly independent vectors in X. Then there is a constant $c > 0$ such that for all $a_1, \dots, a_m \in \mathbb{R}$, $c(|a_1| + \dots + |a_n|) \leq ||a_1x_1 + \dots + a_nx_n||$.*

Proof. Assume the hypothesis. Let $Y = \langle x_1, \dots, x_n\rangle$. Then $(Y, ||\cdot||)$ is a n-dimensional normed vector space, and it is easy to verify that the mapping $||\cdot||_Y : Y \to \mathbb{R}$ defined by

$$\left|\left|\sum_{i=1}^{n} a_i x_i\right|\right|_Y = \sum_{i=1}^{n} |a_i|$$

is a norm on Y. Since Y is finite-dimensional, it follows from fact (3) above that the norms $||\cdot||$ and $||\cdot||_Y$ are equivalent. This immediately yields the conclusion of the lemma. □

In the (crude) proof of [9, Lemma 2.4-1, p. 72], the axiom of choice for denumerable families of non-empty sets of reals is used. However, a suitable adjustment to that proof yields a ZF argument. We only give an outline and we refer the reader to [9] for the rest of the details. We claim that if the formula

$$\exists c > 0,\ \forall c_1, c_2, \dots, c_n \in \mathbb{Q},\ c(|c_1| + |c_2| + \dots + |c_n|) \leq ||c_1x_1 + c_2x_2 + \dots + c_nx_n|| \tag{1}$$

holds, then the conclusion of Lemma 3.4 follows. Assume that formula (1) holds, so let $c > 0$ be a witness of the validity of (1). Assume $a_1, a_2, \dots, a_n$ are in $\mathbb{R}$. Let ε be an arbitrary positive real number. Then

$$c(|a_1| + |a_2| + \dots + |a_n|) \leq ||a_1x_1 + a_2x_2 + \dots + a_nx_n|| + \varepsilon. \tag{2}$$

Indeed, choose rational numbers c_i, $1 \leq i \leq n$ so that for $1 \leq i \leq n$,

$$|a_i - c_i| < \min\left(\frac{\varepsilon}{2(||x_1|| + ||x_2|| + \dots + ||x_n||)}, \frac{\varepsilon}{2cn}\right).$$

Using the triangle inequality of $|\cdot|$ and $||\cdot||$, it can be easily verified that

$$c(|a_1| + |a_2| + \dots + |a_n|) \leq c(|c_1| + |c_2| + \dots + |c_n|) + \frac{\varepsilon}{2} \tag{3}$$

and

$$||c_1x_1 + c_2x_2 + \dots + c_nx_n|| \leq ||a_1x_1 + a_2x_2 + \dots + a_nx_n|| + \frac{\varepsilon}{2}. \tag{4}$$

Combining formulas (1), (3) and (4) gives us formula (2). So in order to derive the conclusion of Lemma 3.4, it suffices to show that formula (1) holds. To this end, one can proceed as in the proof of [9, Lemma 2.4-1] except that no form of AC is required, since the choices that need to be made will be, for each $k \in \omega\backslash\{0\}$, from a set of n-element sequences $(b_1^{(k)}, \ldots, b_n^{(k)})$ of rationals such that $\sum_{i=1}^{n} |b_i^{(k)}| = 1$ and $||b_1^{(k)} x_1 + \cdots + b_n^{(k)} x_n|| < \frac{1}{k}$.

Lemma 3.5 (ZF) *Let $(X, ||\cdot||)$ be a normed vector space. Then the following hold:*
(i) If Y is a finite-dimensional vector subspace of X, then Y is closed in X.
(ii) If Z is a proper vector subspace of X, then $\mathrm{int}(Z) = \varnothing$ (i.e., Z has empty interior)

P r o o f. (i) Assume that Y is finite-dimensional. Then $Y = \langle y_0, \ldots, y_n\rangle$, where $\{y_0, \ldots, y_n\}$ is a basis for Y. We shall show that $\overline{Y} = Y$. To this end, let $y \in \overline{Y}$ and, towards a proof by contradiction, assume that $y \notin Y$. Then the vectors $y, y_0, \ldots, y_n$ are linearly independent. Further, since $y \in \overline{Y}$, we have that $d(y, Y) = \inf\{||y - z|| : z \in Y\} = 0$. By Lemma 3.4, there is a constant $c > 0$ such that for all $a_0, \ldots, a_n \in \mathbb{R}$,

$$||y + (-a_0)y_0 + \cdots + (-a_n)y_n|| \geq c(1 + |a_0| + \cdots + |a_n|) \geq c > 0. \tag{5}$$

Since $Y = \langle y_0, \ldots, y_n\rangle$, it follows from (5) that $d(y, Y) \geq c > 0$, a contradiction. Thus, $y \in Y$ and Y is closed as required.

(ii) This is a well-known ZF-fact. □

We should like to draw the reader's attention here to the fact that the statement "if (X, d) is a metric space and if Y is a complete subspace of X, then Y is closed in X" is not a theorem of ZF. Indeed, the latter statement lies in strength between $\mathsf{AC}^{\aleph_0}$ and $\mathsf{AC}^{\aleph_0}_{\mathrm{fin}}$. For the fact that $\mathsf{AC}^{\aleph_0}$ implies the above statement, the reader is referred to any standard textbook of topology, pin-pointing easily the use of $\mathsf{AC}^{\aleph_0}$. To see that the above statement implies $\mathsf{AC}^{\aleph_0}_{\mathrm{fin}}$, which is equivalent to "every denumerable family of non-empty finite sets has an infinite subfamily with a choice function" (cf. [6]), let $\mathcal{A} = \{A_i : i \in \omega\}$ be a denumerable family of non-empty finite sets, which, without loss of generality, we assume that it is disjoint. Towards a proof by contradiction, assume that $\mathcal{A}$ has no infinite subfamily with a choice function. Let ∞ be an element not in $\bigcup \mathcal{A}$, and also let $X = \bigcup \mathcal{A} \cup \{\infty\}$. Define a map $d : X \times X \to \mathbb{R}$ by requiring

$$d(x, y) = d(y, x) = \begin{cases} 0, & \text{if } x = y, \\ \frac{1}{i+1}, & \text{if } x \neq y \text{ and } x, y \in A_i \cup \{\infty\}, \\ \max\{\frac{1}{i+1}, \frac{1}{j+1}\}, & \text{if } x \in A_i, y \in A_j, \text{ and } i \neq j. \end{cases}$$

It is easy to verify that d is a metric on X, (X, d) is compact, and $Y = \bigcup \mathcal{A}$ is complete, since every Cauchy sequence of elements of Y is eventually constant (since $\mathcal{A}$ has no partial choice function), thus converges to an element of Y, hence Y is complete. However, Y is clearly not closed in X (since it does not contain its accumulation point, namely ∞).

We are now ready to prove that, in ZF, a normed vector space X is finite-dimensional if and only if its closed unit ball $B_X = \{x \in X : ||x|| \leq 1\}$ is compact.

Theorem 3.6 (ZF) *A normed real vector space $(X, ||\cdot||)$ is finite-dimensional if and only if its closed unit ball B_X is compact.*

P r o o f. Let $(X, ||\cdot||)$ be a normed vector space.

($\to$) Assume that X is finite-dimensional. Since B_X is closed and bounded, it follows by Corollary 3.2 that B_X is compact.

($\leftarrow$) Assume that B_X is compact. Toward a proof by contradiction, suppose X is not finite-dimensional. For each $x \in X$ and $\varepsilon > 0$, let $N(x, \varepsilon) = \{y \in X : ||y - x|| < \varepsilon\}$ then the set $\mathcal{C} = \{N(x, \frac{1}{4}) : x \in B_X\}$ is an open cover of B_X. By the compactness assumption, $\mathcal{C}$ has a finite subcover $N(x_1, \frac{1}{4}), N(x_2, \frac{1}{4}), \ldots, N(x_k, \frac{1}{4})$ (of B_X). Since X is not finite-dimensional, $\langle x_1, \ldots, x_k\rangle$ is a proper subspace of X and, by Lemma 3.5(i), $\langle x_1, \ldots, x_k\rangle$ is closed in X. By Riesz's Lemma 3.3, there is a $z \in X$ such that $||z|| = 1$ and $\inf\{||z - y|| : y \in \langle x_1, \ldots, x_k\rangle\} \geq \frac{1}{2}$. It follows that $z \in B_X$ and that $z \notin \bigcup_{i=1}^{n} N(x_i, \frac{1}{4})$, contradicting the fact that $\{N(x_i, \frac{1}{4}) : 1 \leq i \leq k\}$ covers B_X. This completes the proof of ($\leftarrow$) and of the theorem. □

We close this section with a ZF-proof of Mazur's Lemma.

Lemma 3.7 *(i) (Mazur; ZF) Let X be an infinite-dimensional normed space, let Y be a finite-dimensional vector subspace of X, and let $\varepsilon > 0$. Then there is a unit vector $x \in X\backslash Y$ such that $||y|| \leq (1 + \varepsilon)||y + \alpha x||$*

for all $y \in Y$ and all scalars α. Furthermore, the linear projection $p : Y \oplus \langle x \rangle \to Y$ defined by $p(y + \alpha x) = y$, where $y \in Y$ and $\alpha \in \mathbb{R}$, is continuous and $||p|| \leq 1 + \varepsilon$.

(ii) (ZF) *Let X be an infinite-dimensional normed space which is separable, let D be a countable dense subset of the unit sphere $S_X = \{x \in X : ||x|| = 1\}$, let Y be a finite-dimensional vector subspace of X, and also let $\varepsilon > 0$. Then there is an element $x \in D\backslash Y$ such that $||y|| \leq (1 + \varepsilon)||y + \alpha x||$ for all $y \in Y$ and all scalars α.*

Proof. (i) We first note that the proof of the following consequence of HB,

$$\text{if } W \text{ is a non-trivial normed vector space and } x_0 \in W\backslash\{0\}, \text{ then there exists a bounded linear functional } \tilde{f} : W \to \mathbb{R} \text{ such that } ||\tilde{f}||_{W^*} = 1 \text{ and } \tilde{f}(x_0) = ||x_0||_W, \tag{$*$}$$

does not require any choice if W is finite-dimensional (cf. the proof of [16, Theorem 3.2 & its Corollary, pp. 57–59]). Assume the hypotheses and without loss of generality assume that $0 < \varepsilon < 1$. It suffices to show that there exists some unit vector $x \in X$ such that $1 \leq (1 + \varepsilon)||y + \alpha x||$ for all $y \in Y$ with $||y|| = 1$. Since Y is finite-dimensional, it follows (from Theorem 3.6) that the closed unit ball B_Y of Y is compact. We may thus choose unit vectors $y_1, \ldots, y_n$ in Y so that for each unit vector $y \in Y$, there is an i, $1 \leq i \leq n$ such that $||y - y_i|| < \frac{\varepsilon}{2}$. Since X is infinite-dimensional, we may choose a subspace Z of X such that $Y \subseteq Z$, Z is finite-dimensional and $\dim(Z) > \dim(Y) + n$. For each i, $1 \leq i \leq n$, we apply $(*)$ (the "finite dimension" version) to get a bounded linear functional $x_i^* \in Z^*$ such that $||x_i^*|| = 1$ and $x_i^*(y_i) = 1$.

Let $\{z_1, z_2, \ldots, z_k\}$ be a basis for Z. Then an element $z = a_1z_1 + a_2z_2 + \cdots + a_kz_k$ of Z is in $\bigcap_{i=1}^n \mathrm{Ker}(x_i^*)$ if and only if

$$a_1x_i^*(z_1) + a_2x_i^*(z_2) + \cdots + a_kx_i^*(z_k) = 0 \tag{6}$$

for $1 \leq i \leq n$. Thinking of this as a homogeneous linear system of n equations in the k unknowns $a_1, a_2, \ldots, a_k$, we know that the dimension of the solution space S (in $\mathbb{R}^k$) is greater than or equal to $k - n$, and by our assumption $k - n = \dim(Z) - n > \dim(Y)$. Since the map $a_1z_1 + \cdots + a_kz_k \mapsto (a_1, \ldots, a_k)$ is an isomorphism from Z to $\mathbb{R}^k$ which maps $\bigcap_{i=1}^n \mathrm{Ker}(x_i^*)$ onto S we conclude that $\dim(\bigcap_{i=1}^n \mathrm{Ker}(x_i^*)) > \dim(Y)$. We may therefore choose $z \in \bigcap_{i=1}^n \mathrm{Ker}(x_i^*)$ such that $z \notin Y$. Letting $x = \frac{1}{||z||} \cdot z$, we have that $x \in \bigcap_{i=1}^n \mathrm{Ker}(x_i^*)$ and $||x|| = 1$. Now, if α is any scalar and $y \in Y$ satisfies $||y|| = 1$, then choose $1 \leq i \leq n$ with $||y - y_i|| < \frac{\varepsilon}{2}$. We have that

$$||y + \alpha x|| \geq ||y_i + \alpha x|| - ||y - y_i|| \geq ||y_i + \alpha x|| - \frac{\varepsilon}{2} \geq x_i^*(y_i + \alpha x) - \frac{\varepsilon}{2} = 1 - \frac{\varepsilon}{2} \geq \frac{1}{1 + \varepsilon}.$$

Therefore, $1 \leq (1 + \varepsilon)||y + \alpha x||$ for all unit vectors $y \in Y$, as desired.

For the second assertion of the lemma, firstly note that p is clearly linear, and secondly, for all $y \in Y$ and $\alpha \in \mathbb{R}$ we have $||p(y + \alpha x)|| = ||y|| \leq (1 + \varepsilon)||y + \alpha x||$, hence p is bounded, thus continuous, and $||p|| \leq 1 + \varepsilon$. This completes the proof of (i).

(ii) Assume the hypotheses. Firstly, note that S_X is separable without using any form of choice (if $U = \{u_n : n \in \omega\}$ is a countable dense subset of X, then $D = \{\frac{1}{||u_n||}u_n : n \in \omega\}$ is a countable dense subset of S_X). Since Y^c is open (Y is closed being finite-dimensional—cf. Lemma 3.5(i)), it easily follows from the proof of part (i) that the set

$$O_{Y,\varepsilon} = \{x \in S_X : (\forall y \in Y)(\forall \alpha \in \mathbb{R})(||y|| \leq (1 + \varepsilon)||y + \alpha x||)\}$$

has a non-empty interior in S_X. Thus, $O_{Y,\varepsilon} \cap D \neq \varnothing$, and any element x of the latter intersection satisfies the conclusion of (ii). This completes the proof of (ii) and of the lemma. □

We end this section with a lemma which will be useful in the next section (and which is provable in ZF).

Lemma 3.8 *If X is a vector space over $\mathbb{R}$ with a linearly ordered basis B, then for every linearly independent subset C of X, $|C| \leq |\omega \times [B]^{<\omega}|$. In particular, if B is well ordered and infinite then $|C| \leq |B|$.*

Proof. This is clear if B is finite so we assume that B is an infinite and (strictly) linearly ordered by $\prec$. Assume that C is a linearly independent subset of X. Then we have that for all $y \in C$, there is is a unique pair $((a_1, \ldots, a_n), \{b_1, \ldots, bn\})$ for which

1. $n \in \omega\backslash\{0\}$,
2. $(a_1, \ldots, a_n) \in (R\backslash\{0\})^n$,

3. $b_1 \prec \dots \prec b_n$ are in B, and
4. $y = a_1 b_1 + \dots + a_n b_n$.

For each $Q = \{b_1, \dots, b_n\} \in [B]^{<\omega}$ (assuming $b_1 \prec \dots \prec b_n$) let

$$R_Q = \{(a_1, \dots, a_n) : \text{for all } i \text{ such that } 1 \le i \le n,\ a_i \neq 0 \\ \text{and there is a } y \in C \text{ such that } y = a_1 b_1 + \dots + a_n b_n\}$$

and let

$$C_Q = \{y : \exists (a_1, \dots, a_n) \in R_Q \text{ such that } y = a_1 b_1 + \dots + a_n b_n\}.$$

Since C_Q is a linearly independent set and $C_Q \subseteq \langle b_1, \dots, b_n\rangle$ there are no more than n vectors in C_Q. Further, there is a definable one-to-one function from C_Q onto R_Q, namely $h_Q : y \mapsto (a_1, \dots, a_n)$ (if $y = a_1 b_1 + \dots + a_n b_n$). This means that R_Q is finite, say $|R_Q| = k \in \omega$. Using the lexicographic ordering on R_Q (as a subset of $\mathbb{R}^n$) there is a unique order preserving function g_Q from R_Q onto k. This gives us a definable one-to-one function $g_Q \circ h_Q$ from C_Q onto k. Each element of C is in exactly one C_Q and therefore we obtain a one-to-one function from C into $\omega \times [B]^{<\omega}$. (Namely $y \mapsto (g_Q(h_Q(y)), Q)$ where Q is the unique element of $[B]^{<\omega}$ for which $y \in C_Q$.) Thus, $|C| \le |\omega \times [B]^{<\omega}|$ as required. □

4 On the existence of Hamel bases of cardinality $< 2^{\aleph_0}$ for infinite-dimensional Banach spaces without choice

We begin with the proof that the statement "every infinite-dimensional Banach space has a linearly orderable Hamel basis" is a strong choice axiom. Moreover, we show that if in the previous statement we replace "linearly orderable" by well-orderable", then the statement "every infinite-dimensional Banach space has a well-orderable Hamel basis" is equivalent to the full AC.

Theorem 4.1 *(i) "Every infinite-dimensional Banach space has a linearly orderable Hamel basis" implies* OP. *Thus, "every infinite-dimensional Banach space has a linearly orderable Hamel basis" is not provable in* ZF.
(ii) "Every infinite-dimensional Banach space has a well-orderable Hamel basis" implies AC.

P r o o f. (i) Assume the hypothesis and let I be an infinite set. Then the infinite-dimensional Banach space $\ell^2(I)$ has a linearly orderable Hamel basis B. It follows that $\omega \times [B]^{<\omega}$ is linearly orderable (e.g., lexicographically). By Lemma 3.8 (letting C be the set of characteristic functions of singletons in I) we conclude that C is linearly orderable and hence I is linearly orderable. Therefore OP holds.

For the second assertion, any ZF-model in which OP is false, e.g., Pincus' Model $\mathcal{M}4$ in [6], satisfies (from the first assertion) "there exists an infinite-dimensional Banach space with no linearly orderable Hamel basis".

(ii) Assume the hypotheses and let I be an infinite set. Then the infinite-dimensional Banach space $\ell^2(I)$ has a well-orderable Hamel basis B. As in the proof of (i), we let C be the set of characteristic functions of singletons in I and use Lemma 3.8 to conclude that C and therefore I is well orderable. □

As it will be apparent from the forthcoming results, in ZFC, no infinite-dimensional Banach space can have a Hamel basis of cardinality $< 2^{\aleph_0}$. However, the latter proposition *can not be proved from the* ZF *axioms alone*, as we clarify in the subsequent theorem. Thus, some form of choice is necessarily needed for the proof. We shall prove in the sequel that the weak choice principle $\mathsf{AC}_{\mathbb{R}}$, equivalently "$\mathbb{R}$ can be well-ordered", suffices for the proof.

Theorem 4.2 *It is relatively consistent with* ZF *that there exists an infinite-dimensional Banach space which has a Hamel basis of cardinality* $< 2^{\aleph_0}$. *In particular, "no infinite-dimensional Banach space has a Hamel basis of cardinality* $< 2^{\aleph_0}$*" is false in the Basic Cohen Model* (*Model* $\mathcal{M}1$ *in* [6]), *hence* BPI *does not imply the latter statement in* ZF.

P r o o f. Let A be the set of the countably many added generic reals. It is known (cf. [6, 8]) that A is Dedekind-finite in $\mathcal{M}1$, thus $|A| < 2^{\aleph_0}$. Consider the Hilbert space $\ell^2(A)$. From the definition of $\ell^2(A)$ and the fact that A is Dedekind-finite in $\mathcal{M}1$, we have that for all $f \in \ell^2(A)$, the support $s(f) = \{a \in A : f(a) \neq 0\}$ of

f is finite. Thus, the set $B = \{\chi_{\{a\}} : a \in A\}$, where $\chi_{\{a\}}$ is the characteristic function of $\{a\}$, is a Hamel basis for $\ell^2(A)$. Clearly, $|B| = |A| < 2^{\aleph_0}$, hence $\ell^2(A)$ is an infinite-dimensional Banach space having a Hamel basis of cardinality less than $2^{\aleph_0}$. It follows that the statement "no infinite-dimensional Banach space has a Hamel basis of cardinality $< 2^{\aleph_0}$" is false in the Basic Cohen Model. The last assertion of the theorem follows from the fact that BPI is true in $\mathcal{M}1$ (cf. [6]). This completes the proof of the theorem. □

Although Theorem 4.3 below can be obtained as an immediate consequence of the forthcoming Theorem 4.8(ii) ("In ZF, no infinite-dimensional Banach space has a well-orderable Hamel basis of cardinality $< 2^{\aleph_0}$"), we shall include its proof here since, on one hand, it will provide us a useful tool (Claim 4.4) for the establishment of Lemma 4.7 and Theorem 4.8 and, on the other hand, it is interesting in its own right. For the same reasons and in order to exploit further ideas we shall also include the related Remark 4.5.

Theorem 4.3 (ZF) *No infinite-dimensional Banach space has a denumerable Hamel basis.*

Proof. By way of a contradiction, we assume that there exists an infinite-dimensional Banach space X with a denumerable Hamel basis, say $E = \{e_n : n \in \omega\}$. □

Claim 4.4 *X is a separable space.*

Proof. Let $D = \{\sum_{k=0}^{n} q_k e_k : n \in \omega, q_k \in \mathbb{Q}\}$. It is clear that D is denumerable. We assert that D is dense in X. Fix $x \in X$ and $\varepsilon > 0$. Since E is a Hamel basis for X, there exist $a_0, a_1, \ldots, a_m \in \mathbb{R}$ such that $x = \sum_{i=0}^{m} a_i e_i$. For each $i = 0, 1, \ldots, m$ let $q_i \in \mathbb{Q}$ such that $|q_i - a_i| < \frac{\varepsilon}{2(m+1)||e_i||}$, and let $y = \sum_{i=0}^{m} q_i e_i$. Then $y \in D$ and we have that

$$||x - y|| = ||\sum_{i=0}^{m}(a_i - q_i)e_i|| \leq \sum_{i=0}^{m} |a_i - q_i| \cdot ||e_i|| < (m+1) \cdot \frac{\varepsilon}{2(m+1)} = \frac{\varepsilon}{2} < \varepsilon.$$

Therefore, the open ball $B(x, \varepsilon)$ meets D in a non-empty set and D is dense in X as required. □

Since X is separable, it is also second countable, and since X is complete, we have that X is Baire without invoking any form of choice (since all that is needed in the usual proof in order to have X *effectively* Baire is a well-orderable base for its topology). For each $n \in \omega$, let Y_n be the linear span of $\{e_0, e_1, \ldots, e_n\}$. By Lemma 3.5 we have that for each $n \in \omega$, Y_n is a closed nowhere dense set. Moreover, $X = \bigcup\{Y_n : n \in \omega\}$. This contradicts the fact that X is Baire and completes the proof of the theorem.

Remark 4.5 It should be noted here that the statement "no infinite-dimensional Banach space has a denumerable Hamel basis" has occupied the interest of some researchers in the past. In fact, there are witnesses in the literature that vividly show the effort to prove the above statement casting off the Baire Category Theorem BCT. One such luminous example is a proof by Bauer and Benner (cf. [2]) in 1971. The authors, in [2], mention "This paper presents an elementary proof which does not use the category theory. ... Textbooks either omit the result or defer it until after the category theorem is proved."

The proof by Bauer and Benner uses Riesz's ZF-result (cf. Lemma 3.3) reiteratively within an inductive argument in order for a suitable sequence $(x_n)_{n\in\omega\setminus\{0\}}$ (of elements of an infinite-dimensional Banach space X) to be constructed, and as the authors assert, it provides a path which avoids BCT. However, the avoidance of BCT in their proof is *deceptive*. This is because, although the result of BCT does not seem to be used anywhere in the proof, the authors implicitly use the principle of dependent choices DC in order to construct the sequence $(x_n)_{n\in\omega\setminus\{0\}}$ in the first paragraph of their proof. Since DC is logically equivalent to BCT, it is apparent that BCT has been used in the proof. However, it is worth noting that Bauer and Benner's proof can indeed be adjusted in order to truly avoid BCT. Below, we provide in detail the argument which fairly modifies the one in [2] in order to become choice free.

Toward a proof by contradiction, we assume that there exists an infinite-dimensional Banach space X with a denumerable Hamel basis, say $E = \{e_n : n \in \omega\}$. For each $n \in \omega$, let $x_n = \frac{1}{||e_n||} e_n$. It is clear that $F = \{x_n : n \in \omega\}$ is a Hamel basis for X and that $||x_n|| = 1$ for all $n \in \omega$. For each $n \in \omega$, let $Y_n = \langle x_0, \ldots, x_n\rangle$ and let $a_{n+1} = \inf\{||x_{n+1} - y|| : y \in Y_n\}$. Note that for all $n \in \omega$, $a_{n+1} \leq ||x_{n+1}|| = 1$ and $a_{n+1} > 0$, otherwise $d(x_{n+1}, Y_n) = 0$, hence $x_{n+1} \in \overline{Y_n} = Y_n$, a contradiction since F is a Hamel basis for X. (Recall here the ZF-result of Lemma 3.5(i).) Let $b_0 = b_1 = \dfrac{a_1}{3}$, and for $i \in \omega\setminus\{0, 1\}$, let $b_i = \dfrac{a_1 \cdot \ldots \cdot a_{i-1}}{3^{i-1}}$. Since the series $\sum_{i=0}^{\infty} b_i$ converges, it can be easily verified that the sequence of the partial sums of the series $\sum_{i=0}^{\infty} b_i x_i$ is Cauchy, thus converges in X. Let $y = \sum_{i=0}^{\infty} b_i x_i$ be the sum of the series $\sum_{i=0}^{\infty} b_i x_i$ in X. Since F is a Hamel basis for X, there exist $n \in \omega \backslash \{0\}$ and scalars $\lambda_0, \lambda_1, \ldots, \lambda_n \in \mathbb{R}$ so that $y = \sum_{i=0}^{n} \lambda_i x_i$ (some of the λ_i's may be zero). Thus,

$$\sum_{i=0}^{n} (\lambda_i - b_i) x_i - b_{n+1} x_{n+1} - \sum_{i=n+2}^{\infty} b_i x_i = 0,$$

and consequently

$$|| \sum_{i=0}^{n} (\lambda_i - b_i) x_i - b_{n+1} x_{n+1} || = || \sum_{i=n+2}^{\infty} b_i x_i ||,$$

or

$$\frac{a_1 \cdot \ldots \cdot a_n}{3^n} || \sum_{i=0}^{n} \frac{3^n}{a_1 \cdot \ldots \cdot a_n} (\lambda_i - b_i) x_i - x_{n+1} || = || \sum_{i=n+2}^{\infty} b_i x_i ||.$$

Since $\sum_{i=0}^{n} \dfrac{3^n}{a_1 \cdot \ldots \cdot a_n} (\lambda_i - b_i) x_i \in Y_n$, it follows that

$$|| \sum_{i=0}^{n} \frac{3^n}{a_1 \cdot \ldots \cdot a_n} (\lambda_i - b_i) x_i - x_{n+1} || \geq a_{n+1},$$

hence

$$\frac{a_1 \cdot \ldots \cdot a_n}{3^n} || \sum_{i=0}^{n} \frac{3^n}{a_1 \cdot \ldots \cdot a_n} (\lambda_i - b_i) x_i - x_{n+1} || \geq \frac{a_1 \cdot \ldots \cdot a_n \cdot a_{n+1}}{3^n}.$$

Letting $z = \dfrac{a_1 \cdot \ldots \cdot a_n \cdot a_{n+1}}{3^n}$, we have that $z > 0$ and

$$z \leq || \sum_{i=n+2}^{\infty} b_i x_i || \leq \sum_{i=n+2}^{\infty} b_i ||x_i|| = \sum_{i=n+2}^{\infty} b_i \leq z \left(\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots \right) = \frac{z}{2}$$

(recall that $0 < a_n \leq 1$ for all $n \in \omega \backslash \{0\}$). Thus, we have reached a contradiction, finishing the proof.

Another proof of "no infinite-dimensional Banach space has a denumerable Hamel basis" can also be derived from an argument by Lacey in his paper [11] from 1973, where he provides an elementary ZFC-proof that the cardinality of a Hamel basis of an infinite-dimensional separable Banach space is $2^{\aleph_0}$. We shall use ideas from [11] in the proof of the forthcoming Theorem 4.8.

In striking contrast with Theorem 4.3, the statement "no infinite-dimensional Banach space has a Hamel basis which can be written as a denumerable union of finite sets" is unprovable in set theory without the Axiom of Choice. In fact, it turns out that the aforementioned statement is equivalent to a well-known weak choice principle, namely $\mathsf{AC}_{\mathrm{fin}}^{\aleph_0}$. Indeed, we have the following result.

Theorem 4.6 *The following are equivalent in* ZF*:*

1. $\mathsf{AC}_{\mathrm{fin}}^{\aleph_0}$.
2. *No infinite-dimensional Banach space has a Hamel basis which can be written as a denumerable union of finite sets.*

Proof. (1) → (2) Assume $\mathsf{AC}_{\mathrm{fin}}^{\aleph_0}$. If there exists an infinite-dimensional Banach space X with a Hamel basis B that can be written as a denumerable union of finite sets, then by our assumption B is denumerable. This contradicts the result of Theorem 4.3 that no infinite-dimensional Banach space has a denumerable Hamel basis.

(2) → (1) Assume (2). Towards a proof by contradiction assume that there exists a denumerable disjoint family $\mathcal{A} = \{A_n : n \in \omega\}$ of non-empty finite sets, having no partial choice function. Consider the Hilbert space $\ell^2(A)$, where $A = \bigcup \mathcal{A}$. Then for every $f \in \ell^2(A)$, $s(f) = \{x \in A : f(x) \neq 0\}$ (the support of f) is finite. Thus,

$B = \bigcup\{B_n : n \in \omega\}$, where $B_n = \{\chi_{\{a\}} : a \in A_n\}$, is a Hamel basis for $\ell^2(A)$. This is a contradiction, finishing the proof of the implication and of the theorem. □

The next natural step is to replace "denumerable Hamel basis" in the statement "no infinite-dimensional Banach space has a denumerable Hamel basis" by "well-orderable Hamel basis of cardinality $< 2^{\aleph_0}$". As it will be shown in Theorem 4.8(ii), the resulting statement is still a theorem of ZF, but (surprisingly) the argument for the establishment of the result becomes much more involved than the one given for Theorem 4.3. In Theorem 4.8, we also establish that no infinite-dimensional Banach space has a Hamel basis of cardinality $< 2^{\aleph_0}$, presenting two proofs both of which use ideas from [11]. However, the arguments used in [11] are conducted within ZFC, whereas our proofs are carried out in the strictly weaker axiomatic system ZF+ "$\mathbb{R}$ can be well-ordered".

For use in 'Proof A' of the subsequent Theorem 4.8(i), we shall first establish a lemma (Lemma 4.7 below). The reader will soon realize that the certain sequence $(x_n)_{n\in\omega}$ of elements of an infinite-dimensional normed space X which is constructed in the proof of the lemma is apparently related to the notion of a a *Schauder basis* (or simply a *basis*) $(x_n)_{n\in\omega}$ in a Banach space X. Since Mazur's Lemma (Lemma 3.7) shall play a key role in the proof of Lemma 4.7, we recall here some known facts concerning Schauder bases so that the reader obtains a clear picture of the relation of the result of Lemma 4.7 with the aforementioned notion. So, suppose that $(x_n)_{n\in\omega}$ is a basis in a Banach space X, and for each vector $x \in X$, let $x = \sum_{n=0}^{\infty} a_n x_n$ be its unique series representation with respect to the basis $(x_n)_{n\in\omega}$. The *nth-coordinate functional* c_n^* of the basis $(x_n)_{n\in\omega}$ is the linear functional defined by $c_n^*(x) = c_n^*(\sum_{n=0}^{\infty} a_n x_n) = a_n$. (Note that since $(x_n)_{n\in\omega}$ is a Schauder basis, it follows that $(x_n)_{n\in\omega}$ is linearly independent, so for all elements $n, m \in \omega$, we have $c_n^*(x_m) = 1$ if $n = m$ and $c_n^*(x_m) = 0$ if $n \neq m$.)

For every $n \in \omega$, the *projection* $P_n : X \to X$ is defined by

$$P_n(x) = \sum_{i=0}^{n} a_i x_i = \sum_{i=0}^{n} c_i^*(x) x_i.$$

It is known (cf. [1, Theorem 1.37] or [13, Proposition 1.a.2]) that if X is a Banach space (or more generally, a normed vector space) with a basis $(x_n)_{n\in\omega}$ and for each vector $x \in X$, $x = \sum_{n=0}^{\infty} a_n x_n = \sum_{n=0}^{\infty} c_n^*(x) x_n$ is its (unique) series representation with respect to the basis $(x_n)_{n\in\omega}$, then

(i) the function $|||\cdot||| : X \to \mathbb{R}$, defined by $|||x||| = \sup_n ||\sum_{i=0}^{n} a_i x_i||$ is a norm that is equivalent to the norm of X,
(ii) each projection P_n is continuous and $\sup_n ||P_n|| < \infty$, and
(iii) each coordinate functional c_n^* is continuous and $||c_n^*|| \leq \frac{2 \sup_n ||P_n||}{||x_n||}$. (If $||x_n|| = 1$ for all $n \in \omega$, then $\sup_n ||c_n^*|| \leq 2 \sup_n ||P_n|| < \infty$.)

For necessary and sufficient conditions for a sequence $(x_n)_{n\in\omega}$ in a Banach space X to be a Schauder basis, the reader is referred to [1, Theorem 1.41] or [13, Proposition 1.a.3].

Lemma 4.7 *Assume that every Dedekind-finite set of reals is finite. Let X be an infinite-dimensional normed space with a Hamel basis B of cardinality $\leq 2^{\aleph_0}$. Then there is a sequence $(x_n)_{n\in\omega}$ of unit vectors of X such that for every sequence $(a_n)_{n\in\omega}$ of reals if the series $\sum_{n\in\omega} a_n x_n$ converges to 0 (= the additive identity of X), then $a_n = 0$ for every $n \in \omega$.*

Proof. Assume the hypotheses. Since $|B| \leq 2^{\aleph_0}$ and B is infinite, let $(u_n)_{n\in\omega}$ be an injective sequence of elements of B, and also let $W = \langle\{u_n : n \in \omega\}\rangle$. Then W is an infinite-dimensional normed space which is separable, since it has a denumerable Hamel basis, namely $\{u_n : n \in \omega\}$ (recall Claim 4.4 of the proof of Theorem 4.3). Via mathematical induction and using Mazur's Lemma (Lemma 3.7) we shall construct the required sequence of unit vectors of X.

Let $D = \{d_n : n \in \omega\}$ be a countable dense subset of the unit sphere S_W of W. Let $\varepsilon > 0$ and also let $(\varepsilon_n)_{n\in\omega\setminus\{0\}}$ be a sequence of positive real numbers such that $\prod_{n=1}^{\infty}(1 + \varepsilon_n) \leq 1 + \varepsilon$.

For the first step of the induction, we let $x_0 = d_0$.

For the inductive step, assume that we have defined vectors $x_0, x_1, \ldots, x_n$ in D such that for all $0 < i < n + 1$, $||z|| \leq (1 + \varepsilon_i)||z + \alpha x_i||$ for all scalars α and all $z \in \langle x_0, x_1, \ldots, x_{i-1}\rangle$. For $i = 1, \ldots, n$, consider the corresponding linear projections $p_i : \langle x_0, \ldots, x_{i-1}\rangle \oplus \langle x_i\rangle \to \langle x_0, \ldots, x_{i-1}\rangle$. From Lemma 3.7(i), we have that for every i with $0 < i < n + 1$, p_i is continuous and $||p_i|| \leq 1 + \varepsilon_i$.

Now, in view of Lemma 3.7(ii), we may let x_{n+1} be the least element of $D\backslash\langle x_0, \dots, x_n\rangle$ (in the prescribed enumeration of the elements of D given above) such that $||z|| \leq (1 + \varepsilon_{n+1})||z + \alpha x_{n+1}||$ for all scalars α and all $z \in \langle x_0, x_1, \dots, x_n\rangle$. Let $p_{n+1} : \langle x_0, \dots, x_n\rangle \oplus \langle x_{n+1}\rangle \to \langle x_0, \dots, x_n\rangle$ be the corresponding linear projection with $||p_{n+1}|| \leq 1 + \varepsilon_{n+1}$. This completes the inductive step.

Let $V = \langle\{x_n : n \in \omega\}\rangle$. By the construction of $(x_n)_{n\in\omega}$, we have that $\{x_n : n \in \omega\}$ is a Hamel basis for V, thus for every $v \in V$ there is a unique sequence $(a_n)_{n\in\omega}$ of real numbers, which is eventually zero and such that $v = \sum_{n\in\omega} a_n x_n$. Using the fact that the linear projections p_n are bounded, we shall show that the projections $P_n : V \to V$ are also bounded, thus continuous. Indeed, let $n \in \omega$, let $x \in V\backslash\{0\}$, and also let $x = \sum_{n=0}^{\infty} a_n x_n$ be its unique representation (with respect to $(x_n)_{n\in\omega}$), where $(a_n)_{n\in\omega}$ is a sequence of real numbers, which is eventually zero. Let $k \in \omega$ be the largest integer such that $a_k \neq 0$.

If $n \geq k$, then $P_n(x) = x$ (since $a_m = 0$ for all integers $m > k$), thus $||P_n(x)|| = ||x|| \leq (1 + \varepsilon)||x||$.

If $n < k$, then we have

$$\begin{aligned}
||P_n(x)|| &= ||p_{n+1}\left(\sum_{i=0}^{n+1} a_i x_i\right)|| \\
&\leq (1 + \varepsilon_{n+1})||p_{n+2}\left(\sum_{i=0}^{n+2} a_i x_i\right)|| \\
&\leq \cdots \leq (\prod_{i=n+1}^{k-1} (1 + \varepsilon_i))||p_k(x)|| \\
&\leq (\prod_{i=n+1}^{k} (1 + \varepsilon_i))||x|| \\
&\leq (\prod_{i=1}^{k}(1 + \varepsilon_i))||x|| \leq (1 + \varepsilon)||x||.
\end{aligned}$$

Therefore, the linear projection P_n is bounded with $||P_n|| \leq 1 + \varepsilon$, thus P_n is continuous as required.

In order to prove that $(x_n)_{n\in\omega}$ is a sequence which satisfies the conclusion of the lemma, we show next that for each $n \in \omega\backslash\{0\}$, the coordinate functional $c_n^* : V \to \mathbb{R}$ is continuous. To this end, fix $n \in \omega\backslash\{0\}$. Let $x \in V$ and also let $x = \sum_{i=0}^{\infty} a_i x_i = \sum_{i=0}^{\infty} c_i^*(x) x_i$ be its unique representation, where $(a_i)_{i\in\omega}$ is a sequence of real numbers, which is eventually zero. We have

$$|c_n^*(x)| = |a_n| = ||a_n x_n|| = ||P_n(x) - P_{n-1}(x)|| \leq ||P_n(x)|| + ||P_{n-1}(x)|| \leq 2(1 + \varepsilon)||x||.$$

Thus, c_n^* is continuous as required. Now, we show that the constructed sequence $(x_n)_{n\in\omega}$ is the required one. Indeed, let $(a_n)_{n\in\omega}$ be a sequence of real numbers such that $\sum_{n\in\omega} a_n x_n = 0$. Let $(s_n)_{n\in\omega}$ be the sequence of partial sums of the series $\sum_{n\in\omega} a_n x_n$. Note that $s_n = \sum_{i=0}^{n} a_i x_i \in V$ for all $n \in \omega$. Then for every $n \in \omega\backslash\{0\}$, we have $c_n^*(s_m) = a_n$ for all integers $m \geq n$. Since c_n^* is a continuous linear functional, we conclude that for every $n \in \omega$, the sequence $(c_n^*(s_m))_{m\in\omega}$ converges to 0, thus $a_n = 0$ as required. This completes the proof of the lemma. □

Theorem 4.8 *(i) "$\mathbb{R}$ can be well-ordered" implies "No infinite-dimensional Banach space has a Hamel basis of cardinality $< 2^{\aleph_0}$".*

(ii) (ZF) *No infinite-dimensional Banach space has a well-orderable Hamel basis of cardinality $< 2^{\aleph_0}$.*

P r o o f . We first note that (i) follows from (ii). In order to prove (ii) assume towards a proof by contradiction that there exists an infinite-dimensional Banach space X with a well-orderable Hamel basis B such that $|B| < 2^{\aleph_0}$. We obtain a contradiction to Lemma 3.8 by constructing a linearly independent subset Y of X of cardinality $2^{\aleph_0}$.

P r o o f A. By Lemma 4.7, let $(x_n)_{n\in\omega}$ be a sequence of unit vectors in X such that for every sequence $(a_n)_{n\in\omega}$ of reals if the series $\sum_{n\in\omega} a_n x_n$ converges to 0, then $a_n = 0$ for every $n \in \omega$. (Note that the proof of Lemma 4.7 only requires the hypothesis "Every Dedekind-finite set of reals is finite" in order to choose a countably infinite subset of B. In the current proof, this follows from our assumptions that B is infinite and well-orderable.)

Let $\mathcal{A}$ be an almost disjoint family of denumerable subsets of ω of cardinality $2^{\aleph_0}$. (Note that such a family exists without invoking any choice principle; for each irrational number a pick a sequence of rational numbers converging to a. The family of these sequences is a continuum sized almost disjoint family of (necessarily denumerable) subsets of $\mathbb{Q}$.) Pick any real number c in the open interval $(0, 1)$. For each $A \in \mathcal{A}$ consider the following element of X: $t_A = \sum_{n\in A} c^n x_n$. We observe the following:

1. For all $A \in \mathcal{A}$, the series $\sum_{n\in A} c^n x_n$ converges to an element of X. Indeed, the sequence of partial sums of the series $\sum_{n\in A} c^n x_n$ is a Cauchy sequence in X, since $||x_n|| = 1$ for all $n \in \omega$ and the geometric series $\sum_{n\in\omega} c^n$ converges (since $c \in (0, 1)$). Since X is a Banach space, it follows that $\sum_{n\in A} c^n x_n$ converges to an (unique) element in X, which we call t_A.
2. The set $Y = \{t_A : A \in \mathcal{A}\}$ is a linearly independent subset of X: Let $\{A_1, \dots, A_n\}$ be an n-sized subset of $\mathcal{A}$ and let $a_1, \dots, a_n \in \mathbb{R}$ such that $a_1 t_{A_1} + a_2 t_{A_2} + \dots + a_n t_{A_n} = 0$. Since $\mathcal{A}$ is an almost disjoint family of infinite sets, we must have that for each $m \in \{1, \dots, n\}$, $A_m \setminus \bigcup_{k\in\{1,\dots,n\}\setminus\{m\}} A_k$ is infinite, thus for each $m \in \{1, \dots, n\}$ there is a $j_m \in A_m \setminus \bigcup_{k\in\{1,\dots,n\}\setminus\{m\}} A_k$, consequently $a_m c^{j_m} x_{j_m}$ is a term in $a_1 t_{A_1} + a_2 t_{A_2} + \dots + a_n t_{A_n}$. Thus, by the property of $(x_n)_{n\in\omega}$ and the fact that $c^j \neq 0$ for all $j \in \omega$, we infer that $a_m = 0$ for all $m \in \{1, \dots, n\}$.

It follows that $|Y| = 2^{\aleph_0}$ (since for all $A, B \in \mathcal{A}$ with $A \neq B$ we have that $t_A \neq t_B$). The set Y is the required one for obtaining a contradiction to Lemma 3.8.

This completes proof A. □

Proof B. Assume that $\mathbb{R}$ can be well-ordered and again, towards a proof by contradiction, assume that there exists an infinite-dimensional Banach space X with a Hamel basis B such that $|B| < 2^{\aleph_0}$. Since $|X| \leq |\mathbb{R}^{<\omega} \times [B]^{<\omega}|$ (where $\mathbb{R}^{<\omega}$ is the set of all finite sequences of real numbers, cf. also the proof of Lemma 3.8) and $|\mathbb{R}^{<\omega} \times [B]^{<\omega}| = |\mathbb{R} \times B| = 2^{\aleph_0}$ (note that $|\mathbb{R}^{<\omega}| = 2^{\aleph_0}$ in ZF, and $|[B]^{<\omega}| = |B|$ since B is well-orderable), we have that X is well-orderable, so let $\preceq$ be a well-ordering of X.

We shall prove the existence of a sequence $(x_n)_{n\in\omega}$ of unit vectors in X which satisfies the conclusion of Lemma 4.7 without relying on Mazur's Lemma. Then the rest of proof B proceeds as in proof A.

To begin with, we first note that since X is well-ordered, we may effectively define a non-zero element $f_0 \in X^*$. As f_0 is a bounded linear functional, it follows that f_0 is continuous without using any form of choice (we recall that 'bounded' means $|f_0(x)| \leq ||f_0|| \cdot ||x||$ for all $x \in X$, which easily implies continuity). Thus, the kernel $\mathrm{Ker}(f_0)$ of f_0 is a closed subspace of X (since $\mathrm{Ker}(f_0) = f_0^{-1}(\{0\})$). Pick the least (with respect to $\preceq$) element $y_0 \in X$ such that $f_0(y_0) \neq 0$, i.e., $y_0 \notin \mathrm{Ker}(f_0)$. Then $X = \mathrm{Ker}(f_0) \oplus \langle y_0 \rangle$ (i.e., $X = \mathrm{Ker}(f_0) + \langle y_0 \rangle$ and $\mathrm{Ker}(f_0) \cap \langle y_0 \rangle = \{0\}$)—cf. [7, Theorem 3.16]—and so $\mathrm{Ker}(f_0)$ is infinite-dimensional. Let $x_0 = \frac{1}{||y_0||} y_0$, then $||x_0|| = 1$ and $x_0 \notin \mathrm{Ker}(f_0)$.

Construct now a non-zero element $f_1 \in (\mathrm{Ker}(f_0))^*$. Pick the least $y_1 \in \mathrm{Ker}(f_0)$ such that $f_1(y_1) \neq 0$. Then $\mathrm{Ker}(f_0) = \mathrm{Ker}(f_1) \oplus \langle y_1 \rangle$, $\mathrm{Ker}(f_1)$ is closed in $\mathrm{Ker}(f_0)$, thus closed in X, and $\mathrm{Ker}(f_1)$ is infinite-dimensional. Put $x_1 = \frac{1}{||y_1||} y_1$. Then $||x_1|| = 1$ and $x_1 \in \mathrm{Ker}(f_0) \setminus \mathrm{Ker}(f_1)$.

Via mathematical induction we may easily construct a sequence $(x_n)_{n\in\omega}$ of unit vectors of X, a sequence of bounded linear functionals $(f_n)_{n\in\omega}$ such that $f_0 \in X^*$, and for $n > 0$, f_n is a bounded linear functional on $\mathrm{Ker}(f_{n-1})$, hence $\mathrm{Ker}(f_n) \subseteq \mathrm{Ker}(f_{n-1})$ for all positive integers n, and $x_n \in \mathrm{Ker}(f_{n-1}) \setminus \mathrm{Ker}(f_n)$.

It follows that the constructed sequence $(x_n)_{n\in\omega}$ is the required one. Indeed, let $(a_n)_{n\in\omega}$ be any sequence of reals such that $\sum_{n\in\omega} a_n x_n = 0$. We argue by contradiction that $a_n = 0$ for all $n \in \omega$. Let n_0 be the least natural number such that $a_{n_0} \neq 0$. Then $\sum_{n\geq n_0} a_n x_n = 0$, so $a_{n_0} x_{n_0}$ is the limit of the series $\sum_{n>n_0} -(a_n x_n)$, and therefore the sequence of partial sums $(\sum_{n=n_0+1}^{k} a_n x_n)_{k>n_0}$ converges to $-a_{n_0} x_{n_0}$. However, due to the above construction of the sequence (x_n), we have that each term of the sequence $(\sum_{n=n_0+1}^{k} a_n x_n)_{k>n_0}$ is an element of $\mathrm{Ker}(f_{n_0})$ which is closed in X, thus $-a_{n_0} x_{n_0} \in \mathrm{Ker}(f_{n_0})$, and consequently $x_{n_0} \in \mathrm{Ker}(f_{n_0})$. But this is a contradiction, since $x_n \notin \mathrm{Ker}(f_n)$ for all $n \in \omega$. □

From Lemma 4.7 and Theorem 4.8, we easily obtain the following.

Corollary 4.9 *(i)* (ZFA) MC *implies "no infinite-dimensional Banach space has a Hamel basis of cardinality* $< 2^{\aleph_0}$*".*

(ii) "No infinite-dimensional Banach space has a Hamel basis of cardinality $< 2^{\aleph_0}$*" is true in every Fraenkel-Mostowski permutation model of* ZFA.

(iii) Assume that every Dedekind-finite set of reals is finite. If X *is an infinite-dimensional Banach space with a Hamel basis* B *of cardinality* $\leq 2^{\aleph_0}$*, then* X *has a linearly independent subset of cardinality* $2^{\aleph_0}$.

Proof. (i) This follows immediately from the fact that MC implies "$\mathbb{R}$ can be well-ordered" and Theorem 4.8.

(ii) This follows from Theorem 4.8 and the fact that $\mathbb{R}$ is well-orderable in every permutation model of ZFA (cf. [6]).

(iii) Assume the hypothesis. Let X be an infinite Banach space with a Hamel basis B such that $|B| \leq 2^{\aleph_0}$. By Lemma 4.7, there exists a sequence $(x_n)_{n\in\omega}$ of unit vectors of X such that for every sequence $(a_n)_{n\in\omega}$ of reals if the series $\sum_{n\in\omega} a_n x_n$ converges to 0 (= the additive identity of X), then $a_n = 0$ for every $n \in \omega$. Then, from proof A of Theorem 4.8, we have that there exists a linearly independent subset Y of X with cardinality $2^{\aleph_0}$ (cf. the set $Y = \{t_A : A \in \mathcal{A}\}$ in the latter proof). □

We show next that the Hahn-Banach Theorem HB and "no infinite dimensional Banach space has a Hamel basis of cardinality $< 2^{\aleph_0}$" are independent of each other in ZF. Moreover, we shall obtain a sharper result than "no infinite-dimensional Banach space has a Hamel basis of cardinality $< 2^{\aleph_0}$" does not imply HB in ZF, namely we shall construct a ZF-model N such that $N \models \mathsf{DC} + \mathsf{AC}_{\mathbb{R}} + \neg\mathsf{HB}$, hence the desired independence result will then follow from Theorem 4.8. We note that the status of the implication "$\mathsf{DC} + \mathsf{AC}_{\mathbb{R}} \rightarrow \mathsf{HB}$" is stated as unknown in [6], thus our independence result fills the gap in [6].

Theorem 4.10 *(i)* HB *does not imply "no infinite-dimensional Banach space has a Hamel basis of cardinality* $< 2^{\aleph_0}$*" in* ZF.

(ii) $\mathsf{DC} + \mathsf{AC}_{\mathbb{R}}$ *does not imply* HB *in* ZF. *Thus, "no infinite-dimensional Banach space has a Hamel basis of cardinality* $< 2^{\aleph_0}$*" does not imply* HB *in* ZF *either.*

Proof. (i) In the Basic Cohen Model $\mathcal{M}1$ in [6], BPI is true, thus HB is also true. The result now follows from the proof of Theorem 4.2 that in $\mathcal{M}1$ the infinite-dimensional Banach space $\ell^2(A)$ (where A is the set of the countably many added Cohen reals) has a Hamel basis of cardinality $< 2^{\aleph_0}$.

(ii) We start with a countable transitive model M of $\mathsf{ZFC} + \mathsf{CH}$. The partially ordered set of forcing conditions is the set $P = \mathrm{Fn}(\aleph_1 \times \aleph_1, 2, \aleph_1)$, i.e., the set of all partial functions p from $\aleph_1 \times \aleph_1$ into 2 such that $|p| < \aleph_1$, partially ordered by reverse inclusion, i.e., $p \leq q$ if and only if $p \supseteq q$. Let G be a P-generic set over M and let $M[G]$ be the corresponding extension model of M.

Every set $X \subseteq \aleph_1 \times \aleph_1$ induces an order automorphism π_X of $(P, \leq)$ via

$$\pi_X(p)(u, v) = \begin{cases} p(u, v) & \text{if } (u, v) \notin X, \\ 1 - p(u, v) & \text{if } (u, v) \in X, \end{cases}$$

for any $p \in P$. Let G be the group generated by the sets $G_1 = \{\pi_X : X \subseteq \aleph_1 \times \aleph_1\}$ and $G_2 = \{\pi \in \mathrm{Sym}(\aleph_1 \times \aleph_1) : \pi$ moves only elements in finitely many columns$\}$, thus if $\pi \in G_2$ then π is a a one-to-one mapping from $\{n\} \times \aleph_1$ onto $\{m\} \times \aleph_1$ for finitely many ordinals n, m, and π fixes all the other columns pointwise. Every permutation $\pi \in G_2$ induces an order automorphism of $(P, \leq)$ via $\pi(p)(\pi(n, m)) = p(n, m)$. For every countable set $E \subseteq \aleph_1 \times \aleph_1$ let $\mathrm{fix}(E)$ be the subgroup of G generated by $\{\pi_X : X \subseteq \aleph_1 \times \aleph_1$ and $X \cap (\mathrm{dom}(E) \times \aleph_1) = \varnothing\} \cup \{\pi \in G_2 : \pi(e) = e$ for all $e \in E\}$. Let $\mathcal{F}$ be the normal filter generated by $\{\mathrm{fix}(E) : E \in [\aleph_1 \times \aleph_1]^{<\aleph_1}\}$. Let N be the symmetric extension model of M determined by G and $\mathcal{F}$.

For each $n \in \aleph_1$, let (in $M[G]$) $a_n = \{m \in \aleph_1 : \exists p \in G, p(n, m) = 1\}$ and $\delta(a_n) = \{a_n \triangle x : x \in [\aleph_1]^{<\aleph_1}\}$ (i.e., $\delta(a_n)$ is the $\sim$ equivalence class of a_n where $\sim$ is the equivalence relation on $\wp(\aleph_1)$ defined by $x \sim y$ if and only if $|x \triangle y| < \aleph_1$). Both a_n and $\delta(a_n)$ (as well as $\delta(\aleph_1 \backslash a_n)$) have canonical names $\overline{a_n}$ and $\overline{\delta(a_n)}$ ($\overline{\delta(\aleph_1 \backslash a_n)}$) which are hereditarily symmetric, so a_n and $\delta(a_n)$ (as well as $\delta(\aleph_1 \backslash a_n)$) belong to N.

1. Since $\aleph_1$ is a regular cardinal number, it follows by [10, Lemma 6.13, p. 214] that $(P, \leq)$ is a $\aleph_1$-closed poset, thus forcing with P adds no new reals (cf. [10, Theorem 6.14, p. 214]), but it adds new sets of reals. Thus, CH is true in N and consequently $\mathbb{R}$ is well-orderable in the model N. (Note that this implies the existence of a free ultrafilter on ω, hence on $\mathbb{R}$, and thus implies the existence of a 2-valued measure on $\wp(\omega)$.)

2. Since $\aleph_1$ is a regular cardinal number and supports are countable, we also have that DC is true in N (cf. [8, Lemma 8.5, p. 124]).

Now, it is known (cf. [6]) that HB is equivalent to "for every proper ideal $\mathcal{I}$ of elements of a Boolean algebra $\mathcal{B}$, there exists a finitely additive real-valued measure m on $\mathcal{B}$ such that $m(x) = 0$ for all $x \in \mathcal{I}$" (this is [6, Form [52 B]]). So let $\mathcal{I}$ be the ideal of all countable subsets of $\aleph_1$. Towards a proof by contradiction assume that there is a finitely additive real-valued measure m on $\wp(\aleph_1)$ such that $m(x) = 0$ for all $x \in \mathcal{I}$. Let $\dot{m}$ be a HS-name for m with support some countable set $E \subseteq \aleph_1 \times \aleph_1$. We shall show that for every $n \in \aleph_1 \setminus \text{dom}(E)$, $m(a_n) = m(\aleph_1 \setminus a_n)$ and thus by the finite additivity of m we shall have that $m(a_n) = \frac{1}{2}$ for all $n \in \aleph_1 \setminus \text{dom}(E)$ (for $m(\aleph_1) = 1$). So fix $n \in \aleph_1 \setminus \text{dom}(E)$ and assume that $m(a_n) = r$ for some non-negative real r. Then there exists $p \in G$ such that

$$p \Vdash \dot{m}(\overline{a_n}) = \check{r}, \tag{7}$$

where $\overline{a_n}$ is the (HS-) canonical name of a_n. Since $|p| < \aleph_1$, there exists an ordinal $k_0 \in \aleph_1$ such that for all $k \geq k_0$, $(n, k) \notin \text{dom}(p)$. Let $X = \{(n, k) : k \in \aleph_1, k \geq k_0\}$. Since $X \cap (\text{dom}(E) \times \aleph_1) = \varnothing$ we have that $\pi_X \in \text{fix}(E)$, thus $\pi_X(\dot{m}) = \dot{m}$. Furthermore, $\pi_X(p) = p$. Thus, from formula (7) it follows that

$$p \Vdash \dot{m}(\pi_X(\overline{a_n})) = \check{r}. \tag{8}$$

Since $p \in G$ we have, by formula (8), that $m((\pi_X(\overline{a_n}))_G) = r$, thus $m(a_n) = m((\pi_X(\overline{a_n}))_G)$. It is fairly straightforward now to verify that $(\pi_X(\overline{a_n}))_G \in \delta(\aleph_1 \setminus a_n)$, so since m vanishes on countable subsets of $\aleph_1$ it follows that $m((\pi_X(\overline{a_n}))_G) = m(\aleph_1 \setminus a_n)$ (let $W = (\aleph_1 \setminus a_n) \cap (\pi_X(\overline{a_n}))_G$, then $m(\aleph_1 \setminus a_n) = m((\aleph_1 \setminus a_n) \setminus W) + m(W) = 0 + m(W) = m((\pi_X(\overline{a_n}))_G \setminus W) + m(W) = m((\pi_X(\overline{a_n}))_G)$). Therefore $m(a_n) = m(\aleph_1 \setminus a_n)$. Furthermore, note that $a_n \cap (\pi_X(\overline{a_n}))_G \subseteq k_0$, thus $a_n \cap (\pi_X(\overline{a_n}))_G$ is countable.

Therefore, from the above argument we have that there is a forcing condition $q \in G$ such that

$$q \Vdash (\dot{m} \text{ is a measure on } \dot{\wp(\aleph_1)} \text{ vanishing on countable sets}) \wedge (\dot{m}(\overline{a_n}) = \check{\tfrac{1}{2}}) \wedge (\dot{m}(\pi_X(\overline{a_n})) = \check{\tfrac{1}{2}}). \tag{9}$$

Now, let $n' \in \aleph_1 \setminus (\text{dom}(E) \cup \{n\} \cup \text{dom}(E'))$, where E' is a support of $\dot{\wp(\aleph_1)}$. It is fairly easy to find a permutation $\psi \in G_2$ which fixes $E \cup E'$ pointwise (in particular, ψ fixes each column $\{e\} \times \aleph_1$, $e \in \text{dom}(E) \cup \text{dom}(E')$, pointwise), thus fixes $\dot{m}$, and ψ maps the names $\overline{a_n}$ and $\pi_X(\overline{a_n})$ up to a countable set onto disjoint subsets A and B of $\overline{a_{n'}}$. Clearly, A and B are names (cf. [10, Definition 2.5, p. 188]) and furthermore they are hereditarily symmetric.

From formula (9) we may conclude that

$$\psi(q) \Vdash (\dot{m} \text{ is a measure on } \dot{\wp(\aleph_1)} \text{ vanishing on countable sets}) \wedge (\dot{m}(A) = \check{\tfrac{1}{2}}) \wedge (\dot{m}(B) = \check{\tfrac{1}{2}}). \tag{10}$$

From (10) we obtain a contradiction: Indeed, let H be a P-generic filter over M such that $\psi(q) \in H$. Then in $M[H]$ we have that $\dot{m}_H$ is a measure on $\dot{\wp(\aleph_1)}_H$ vanishing on countable subsets of $\aleph_1$, $\dot{m}_H((\overline{a_{n'}})_H) = \frac{1}{2}$ (exactly as in the argument that for all $n \in \aleph_1 \setminus \text{dom}(E)$, $m(a_n) = \frac{1}{2}$ in $M[G]$) $\dot{m}_H(A_H) = \frac{1}{2}$, $\dot{m}_H(B_H) = \frac{1}{2}$, $|A_H \cap B_H| \leq \aleph_0$, and $A_H \cup B_H \subseteq (\overline{a_{n'}})_H$. Let $T = A_H \cap B_H$. By the finite additivity of the measure $\dot{m}_H$ and the fact that $\dot{m}_H(T) = 0$ we have that $\dot{m}_H(A_H \setminus T) = \frac{1}{2}$ and $\dot{m}_H(B_H \setminus T) = \frac{1}{2}$. Therefore, we have

$$1 = \dot{m}_H(A_H \setminus T) + \dot{m}_H(B_H \setminus T) = \dot{m}_H(A_H \triangle B_H) \leq \dot{m}_H((\overline{a_{n'}})_H) = \frac{1}{2},$$

(the last inequality follows from the monotonicity of the measure, i.e., $U \subseteq V$ implies $\dot{m}_H(U) \leq \dot{m}_H(V)$; $\dot{m}_H(V) = \dot{m}_H((V \setminus U) \cup U) = \dot{m}_H(V \setminus U) + \dot{m}_H(U) \geq \dot{m}_H(U)$), and we have reached a contradiction.

Thus, in N, there is no finitely additive real-valued measure m on $\wp(\aleph_1)$ such that $m(x) = 0$ for every countable subset $x \subset \aleph_1$, and HB is false in N as required.

The proof of (ii), as well as of the theorem, is complete. □

5 Infinite-dimensional Banach spaces may be written as denumerable unions of closed proper subspaces or finite-dimensional subspaces in the lack of AC

In this section, we investigate the deductive strength of the statements

a) no infinite-dimensional Banach space can be written as a denumerable union of closed proper subspaces,
b) no infinite-dimensional Banach space can be written as a denumerable union of finite-dimensional subspaces.

It is clear that BCT → (a) → (b). From the subsequent results, we shall derive that (b), hence (a), is not provable in ZF.

Theorem 5.1 *Each of the following statements implies the one beneath it:*

(i) DC*;*
(ii) no infinite-dimensional Banach space can be written as a denumerable union of closed proper subspaces;
(iii) $\mathsf{AC}^{\aleph_0}$*;*
(iv) no infinite-dimensional Banach space can be written as a denumerable union of finite-dimensional subspaces;
(v) $\mathsf{AC}^{\aleph_0}_{\mathsf{fin}}$.

P r o o f . (i) → (ii) This readily follows from the fact that DC is (in ZF) equivalent to BCT and from Lemma 3.5(ii).

(ii) → (iii) Assume that no infinite-dimensional Banach space can be written as a denumerable union of closed proper subspaces. Since $\mathsf{AC}^{\aleph_0}$ is equivalent to its partial version, it suffices to show that every denumerable family of non-empty sets has a partial choice function. Assume on the contrary that there exists a denumerable family $\mathcal{A} = \{A_i : i \in \omega\}$, where $A_i \neq \varnothing$ for all $i \in \omega$, without a partial choice function.

Consider the Hilbert space $H = \ell^2(A)$, where $A = \bigcup \mathcal{A}$. Since $\mathcal{A}$ does not have a partial choice function, it follows that for every $f \in H$, the support $s(f)$ of f is contained in some finite union of A_i's. Thus, $H = \bigcup\{H_n : n \in \omega\}$ where $H_n = \ell^2(A_0 \cup A_1 \cup \ldots \cup A_n)$ (with the intended meaning that H_n consists of all functions $f \in H$ such that for all $m > n$ and all $x \in A_m$, $f(x) = 0$). Then for all $n \in \omega$, H_n is a proper subspace of H, and also, $H_1 \subsetneq H_2 \subsetneq H_3 \subsetneq \ldots$. We assert that H_n is closed in H for every $n \in \omega$. To prove our assertion, we fix a positive integer n and we also let $f \in H \backslash H_n$. We look for a positive real r such that the open ball $B(f, r)$ centered at f and of radius r, does not meet H_n. Since $f \notin H_n$ $(= \ell^2(A_0 \cup A_1 \cup \ldots \cup A_n))$, there exists a natural number $m > n$ and an element $x \in A_m$ such that $f(x) \neq 0$. Let ε be any positive real such that $\varepsilon < (f(x))^2$. Then the open ball $B(f, r) = \{g \in H : ||f - g|| < r\}$, where $r = \sqrt{\varepsilon}$, is such that $B(f, r) \cap H_n = \varnothing$. Assume the contrary and let $g \in B(f, r) \cap H_n$. Then we have that $g(x) = 0$ and

$$||f - g|| = \sqrt{\Big(\sum_{y \in (s(f)\backslash\{x\}) \cup s(g)} (f(y) - g(y))^2\Big) + (f(x))^2} < \sqrt{\varepsilon},$$

or equivalently,

$$\Big(\sum_{y \in (s(f)\backslash\{x\}) \cup s(g)} (f(y) - g(y))^2\Big) + (f(x))^2 < \varepsilon.$$

Since $\varepsilon < (f(x))^2$, the last inequality obviously yields a contradiction. Therefore, $B(f, r) \cap H_n = \varnothing$ and H_n is closed in H as required. Hence, the infinite-dimensional Banach space H can be written as a denumerable union of closed proper subspaces, contradicting our assumption. It follows that the family $\mathcal{A}$ has a partial choice function and $\mathsf{AC}^{\aleph_0}$ holds, finishing the proof.

(iii) → (iv) Assume $\mathsf{AC}^{\aleph_0}$, and towards a proof by contradiction, assume that there is an infinite-dimensional Banach space X which can be written as a denumerable union $\bigcup\{X_n : n \in \omega\}$, where each X_n is a finite-dimensional subspace of X. By $\mathsf{AC}^{\aleph_0}$, pick for each $n \in \omega$, a finite subset Y_n of X_n such that $X_n = \langle Y_n \rangle$. It is clear that $Y = \bigcup\{Y_n : n \in \omega\}$ spans X and since X is infinite-dimensional, it follows that Y is infinite, and in particular, by $\mathsf{AC}^{\aleph_0}$, it is denumerable. Without loss of generality we may also assume that $0 \notin Y$. Now, there are two ways to proceed with the proof. We present them both.

P r o o f 1. Since X is spanned by the denumerable set Y, it follows that X is separable (this can be shown exactly as in the proof of Claim 4.4 of the proof of Theorem 4.3). Thus, X is Baire, without using any form of choice, and we have reached a contradiction, since X is a denumerable union of the closed nowhere dense sets X_n (X_n is closed since it is finite-dimensional and it is nowhere dense since it is a proper subset of X, cf. Lemma 3.5).

P r o o f 2. Using an enumeration of Y, we may construct via an easy mathematical induction a maximal linearly independent subset of Y, say B. Then B is a denumerable Hamel basis for X, which contradicts the result of Theorem 4.3.

(iv) → (v) Assume (iv). In view of Theorem 4.6, it suffices to show that (iv) implies that no infinite-dimensional Banach space has a Hamel basis which can be written as a denumerable union of finite sets. By way of a contradiction, assume that there is an infinite-dimensional Banach space X with a Hamel basis $B = \bigcup\{B_n : n \in \omega\}$, where $|\{B_n : n \in \omega\}| = \aleph_0$ and for each $n \in \omega$, B_n is a non-empty finite subset of X. Then $X = \bigcup\{X_n : n \in \omega\}$, where $X_n = \langle B_0 \cup B_1 \cup \cdots \cup B_n\rangle$. Since each X_n is finite-dimensional, we obtain a contradiction to the assumption of (iv), finishing the proof of the theorem. □

It is clear that BCT implies "every infinite-dimensional Banach space is Baire" implies "no infinite-dimensional Banach space can be written as a denumerable union of closed proper subspaces".

We do not know whether any of the above implications is reversible, or whether $\mathsf{AC}^{\aleph_0}$ implies "no infinite-dimensional Banach space can be written as a denumerable union of closed proper subspaces", or whether the latter functional analytic statement is equivalent to DC, or even if $\mathsf{AC}^{\aleph_0}_{\mathsf{fin}}$ implies "no infinite-dimensional Banach space can be written as a denumerable union of finite-dimensional subspaces". However, we are able to show that the implication "$\mathsf{AC}^{\aleph_0}$ implies (no infinite-dimensional Banach space can be written as a denumerable union of finite-dimensional subspaces)" is not reversible in ZFA set theory, as our subsequent Theorem 5.2 clarifies.

We should also like to point out here that in view of Theorem 5.1 and of the forthcoming Theorem 6.1, it is a striking result (and unexpected) that "no infinite-dimensional Banach space can be written as a denumerable union of closed proper subspaces" implies "for every field F, every infinite-dimensional vector space V over F has a denumerable linearly independent subset".

Theorem 5.2 *(i)* $(\mathsf{CH} + \mathsf{W}_{\aleph_2})$ *implies "no infinite-dimensional Banach space can be written as a denumerable union of finite-dimensional subspaces".*

(ii) "No infinite-dimensional Banach space can be written as a denumerable union of finite-dimensional subspaces" does not imply $\mathsf{AC}^{\aleph_0}$ *in* ZFA.

P r o o f . (i) Assume the hypothesis. In view of the forthcoming Theorem 6.4, we have that under our hypothesis every infinite-dimensional Banach space has a linearly independent subset of cardinality $\geq 2^{\aleph_0}$. In order to complete the proof of (i), we establish the following claim.

Claim 5.3 *"Every infinite-dimensional Banach space has a linearly independent subset of cardinality* $\geq 2^{\aleph_0}$*" implies "no infinite-dimensional Banach space can be written as a denumerable union of finite-dimensional subspaces".*

P r o o f . Assume the hypothesis of the claim and let X be an infinite-dimensional Banach space. By way of a contradiction, assume that $X = \bigcup\{X_n : n \in \omega\}$, where $n \mapsto X_n$ is a bijection and each X_n is a finite-dimensional subspace of X. Let I be a linearly independent subset of X of size $2^{\aleph_0}$. Then $I = \bigcup\{I \cap X_n : n \in \omega\}$, hence for some $n \in \omega$, $I \cap X_n$ must be infinite (otherwise, fixing a linear order on I we should have that I is a countable union of finite well-ordered sets, thus I is countable, which is a contradiction). Since X_n is finite-dimensional and $I \cap X_n$ is an infinite linearly independent set, we have reached a contradiction. This completes the proof. □

(ii) For the independence result, we use the permutation model $\mathcal{N}16$ in [6], but with the restriction that we start with a ground model of $\mathsf{ZFA} + \mathsf{AC} + \mathsf{CH}$. We recall that $\mathcal{N}16$ is constructed by starting with a set A of atoms of cardinality $\aleph_\omega$. The group G is the group of all permutations of A and supports are subsets of the atoms of cardinality less than $\aleph_\omega$. In this model, $\mathsf{AC}^{\aleph_0}$ is false and for every $n \in \omega$, $\mathsf{W}_{\aleph_n}$ is true. Thus, by (i), no infinite-dimensional Banach space can be written as a denumerable union of finite-dimensional subspaces in the model, finishing the proof. □

Below, we provide some further terminology and notation which will be needed for our next result (Theorem 5.5).

1. Let $\mathcal{V} = \{V_i : i \in I\}$ be a family of vector spaces over some field F. The *direct sum* (or *weak direct product*) of the V_i's is $\bigoplus_{i\in I} V_i = \{f \in \prod_{i\in I} V_i : |\{i \in I : f(i) \neq 0\}| < \aleph_0\}$. It is easy to verify that $\bigoplus_{i\in I} V_i$ is a vector space over F with pointwise addition and scalar multiplication. Furthermore, if for each $i \in I$, $||\cdot||_i$ is a norm on V_i, then $||f|| = \sum_{i\in I} ||f(i)||_i$ is a norm on $\bigoplus_{i\in I} V_i$.
2. Let $n \in \mathbb{N}$ and let $X_1, X_2, \ldots, X_n$ be Banach spaces. Then the direct sum vector space $X_1 \oplus X_2 \oplus \cdots \oplus X_n$ is a Banach space under the norm $||(x_1, x_2, \ldots, x_n)|| = \sum_{i=1}^{n} ||x_i||_i$ (cf. [1, pp. 5–6]).
3. $\leq\aleph_0$-$\mathsf{MC}^{\aleph_0}$ is the weak choice principle: *For every denumerable disjoint family $\mathcal{A} = \{A_i : i \in \omega\}$ of non-empty sets, there exists a function f with domain ω such that for each $i \in \omega$, $f(i)$ is a non-empty countable subset of A_i.*

Lemma 5.4 $\leq\aleph_0$-$\mathsf{MC}^{\aleph_0}$ *if and only if* $\leq\aleph_0$-$\mathsf{PMC}^{\aleph_0}$ *(= "for every denumerable disjoint family $\mathcal{A} = \{A_i : i \in \omega\}$ of non-empty sets, there exists an infinite subfamily $\mathcal{B}$ of $\mathcal{A}$ and a function f with domain $\mathcal{B}$ such that for each $B \in \mathcal{B}$, $f(B)$ is a non-empty countable subset of B").*

Proof. (←) Let $\mathcal{A} = \{A_i : i \in \omega\}$ be a denumerable disjoint family of non-empty sets. Let $B_0 = A_0$ and for each positive integer i, let $B_i = \prod_{j\leq i} A_j$. Then the family $\mathcal{B} = \{B_i : i \in \omega\}$ has an infinite subfamily $\mathcal{C} = \{B_{n_i} : i \in \omega\}$, $(n_i)_{i\in\omega}$ a strictly increasing sequence of natural numbers, with a function g such that for all $i \in \omega$, $g(i)$ is a non-empty countable subset of B_{n_i}. On the basis of f and via mathematical induction, one may easily construct a function f satisfying the conclusion of $\leq\aleph_0$-$\mathsf{MC}^{\aleph_0}$ for the given family $\mathcal{A}$. □

Theorem 5.5 *Each of the following statements implies the one beneath it:*

(a) $\mathsf{AC}^{\aleph_0}$*;*
(b) *if $\{(X_i, ||\cdot||_i) : i \in \omega\}$ is a denumerable family of non-trivial Banach spaces, then the direct sum vector space $\bigoplus_{i\in\omega} X_i$ is not a Banach space under the norm $||f|| = \sum_{i\in\omega} ||f(i)||_i$;*
(c) $\leq\aleph_0$-$\mathsf{MC}^{\aleph_0}$.

Proof. (a) → (b) Assume $\mathsf{AC}^{\aleph_0}$ and let $\{X_i : i \in \omega\}$ be a denumerable family of non-trivial Banach spaces. Since $X_i\backslash\{0\} \neq \varnothing$ for all $i \in \omega$, we may pick, via $\mathsf{AC}^{\aleph_0}$, an element $x_i \in X_i\backslash\{0\}$ for each $i \in \omega$. Furthermore, we may assume that $||x_i||_i = \frac{1}{2^i}$ for all $i \in \omega$ (if $||x_i||_i \neq \frac{1}{2^i}$, we may consider the element $x_i' = \frac{1}{2^i||x_i||_i} x_i$). Let $(a_n)_{n\in\omega}$ be the following sequence of elements of $\bigoplus_{i\in\omega} X_i$: For each $n \in \omega$, we let

$$a_n(i) = \begin{cases} x_i & \text{if } i \leq n, \\ 0 & \text{if } i > n. \end{cases}$$

It is not hard to verify that $(a_n)_{n\in\omega}$ is a Cauchy sequence in $\bigoplus_{i\in\omega} X_i$, which does not converge in $\bigoplus_{i\in\omega} X_i$ (otherwise, its limit should have only finitely many non-zero coordinates, which is impossible). Thus, $\bigoplus_{i\in\omega} X_i$ fails to be a Banach space, finishing the proof of the implication.

(b) → (c) Assume the statement in (b). In view of Lemma 5.4, it suffices to show that $\leq\aleph_0$-$\mathsf{PMC}^{\aleph_0}$ holds. Towards a proof by contradiction assume that there exists a denumerable disjoint family $\mathcal{A} = \{A_i : i \in \omega\}$ which has no infinite subfamily satisfying the conclusion of $\leq\aleph_0$-$\mathsf{PMC}^{\aleph_0}$ for $\mathcal{A}$. For each $i \in \omega$, consider the Banach space $X_i = \ell^2(A_i)$. Let $\bigoplus_{i\in\omega} X_i$ be the direct sum vector space endowed with the norm $||(x_i)_{i\in\omega} = \sum_{i\in\omega} ||x_i||_i$. From our hypothesis, we have that $(\bigoplus_{i\in\omega} X_i, ||\cdot||)$ is not a Banach space.

We shall obtain a contradiction by showing that the pair $(\bigoplus_{i\in\omega} X_i, ||\cdot||)$ is a Banach space. To this end, let $(a_n)_{n\in\omega}$ be a Cauchy sequence in $\bigoplus_{i\in\omega} X_i$. Since $\mathcal{A}$ does not have a partial $\leq\aleph_0$-$\mathsf{MC}^{\aleph_0}$ function, it follows that there is an $m \in \omega$ such that for all $n \in \omega$ and all $i > m$, we have that $a_n(i) = 0$ (otherwise, via mathematical induction, and using the definition of $\ell^2(\cdot)$, we may easily construct an infinite subfamily of $\mathcal{A}$ with an $\leq\aleph_0$-$\mathsf{MC}^{\aleph_0}$ function; we take the liberty to leave the details to the interested reader).

But then, the sequence $(a_n)_{n\in\omega}$ can be viewed as a Cauchy sequence in the direct sum $X_1 \oplus X_2 \oplus \cdots \oplus X_m \oplus \{0\} \oplus \{0\} \oplus \cdots$, which is homeomorphic with the finite direct sum Banach space $X_1 \oplus X_2 \oplus \cdots \oplus X_m$ (with the norm $||(x_i)_{i\leq m}|| = \sum_{i\leq m} ||x_i||_i$). It follows that $(a_n)_{n\in\omega}$ converges to an element $a \in X_1 \oplus X_2 \oplus \cdots \oplus X_m \oplus \{0\} \oplus \{0\} \oplus \cdots$, thus converges to an element of $\bigoplus_{i\in\omega} X_i$. Consequently, $(\bigoplus_{i\in\omega} X_i, ||\cdot||)$ is a Banach space, which is a contradiction. It follows that $\mathcal{A}$ has a partial $\leq\aleph_0$-$\mathsf{MC}^{\aleph_0}$ function, hence $\leq\aleph_0$-$\mathsf{MC}^{\aleph_0}$ holds, finishing the proof of the implication and of the theorem. □

6 On the existence of infinite linearly independent sets of certain cardinality in infinite-dimensional Banach spaces

In this section, we shall be concerned with the set-theoretic strength of the statements

(a) every infinite-dimensional Banach space has a linearly independent subset of cardinality $\geq 2^{\aleph_0}$,

(b) every infinite-dimensional Banach space has a denumerable linearly independent subset.

It is clear that (a) → (b). We begin by establishing a result on the set-theoretic strength of the more general statement "for every field F, every infinite-dimensional vector space V over F has a denumerable linearly independent subset".

Theorem 6.1 *Each of the following statements implies the one beneath it:*

(i) $\mathsf{AC}^{\aleph_0}$*;*

(ii) for every field F, every infinite-dimensional vector space V over F has a denumerable linearly independent subset;

(iii) $\mathsf{MC}^{\aleph_0}$.

P r o o f. (i) → (ii) Assume $\mathsf{AC}^{\aleph_0}$. Let F be any field and let X be an infinite-dimensional vector space over F. For each $n \in \omega\backslash\{0\}$, let

$$A_n = \{(x_0, x_1, \ldots, x_n) \in X^{n+1} : x_0 \neq 0, \text{ and for all } 1 \leq i \leq n, x_i \notin \langle x_0, x_1, \ldots, x_{i-1}\rangle\}.$$

Since X is infinite-dimensional, it follows that $A_n \neq \varnothing$ for all $n \in \omega\backslash\{0\}$. Let $\mathcal{A} = \{A_n : n \in \omega\backslash\{0\}\}$ and let, by $\mathsf{AC}^{\aleph_0}$, $f = \{(n, (x_0^{(n)}, x_1^{(n)}, \ldots, x_n^{(n)})) : n \in \omega\backslash\{0\}\}$ be a choice function of $\mathcal{A}$. Note that by the definition of A_n, $\mathrm{ran}(f(n))$ is an $(n+1)$-sized set of linearly independent vectors of X. Let $A = \bigcup\{\mathrm{ran}(f(n)) : n \in \omega\backslash\{0\}\}$. It is clear that A is denumerable. Furthermore, since A has finite sequences of linearly independent vectors of arbitrary finite length, we may construct—via mathematical induction—a denumerable linearly independent subset of X. Indeed, let $y_0 = x_0^{(1)}$. Then y_0 is linearly independent, since $y_0 \neq 0$ (cf. the definition of A_n). Assume that for some $n \in \omega\backslash\{0\}$ we have chosen linearly independent vectors $y_0, y_1, \ldots, y_n \in A$. Since $\dim(\langle y_0, y_1, \ldots, y_n\rangle) = n+1$ and $\mathrm{ran}(f(n+1))$ consists of $n+2$ linearly independent vectors, it follows that there exists an element $x \in \mathrm{ran}(f(n+1))$ which does not belong to $\langle y_0, y_1, \ldots, y_n\rangle$. Let $j_{n+1} = \min\{j : j < n+2 \text{ and } x_j^{(n+1)} \in \mathrm{ran}(f(n+1))\backslash\langle y_0, y_1, \ldots, y_n\rangle\}$. Put $y_{n+1} = x_{j_{n+1}}^{(n+1)}$. This completes the inductive step.

From the inductive construction above, we conclude that $\{y_n : n \in \omega\}$ is a denumerable linearly independent subset of X.

(ii) → (iii) Assume the hypothesis. It suffices to show that every denumerable family of non-empty sets has a partial multiple choice function. To this end, let $\mathcal{X} = \{X_i : i \in \omega\}$ be a denumerable family of non-empty sets which, without loss of generality, we assume that it is disjoint. Let $X = \bigcup \mathcal{X}$ and let F be any field which is disjoint from X. Let $F(X)$ be the field of all rational functions with indeterminates from X and coefficients in F. (Every element $u \in F(X)$ is of the form $\frac{p_1+\ldots+p_n}{q_1+\ldots+q_m}$, where p_i and q_i are monomials, i.e., of the form $a \cdot x_1^{n_1} \cdot x_2^{n_2} \cdot \ldots x_k^{n_k}$ where $a \in F$ and the x_r's belong to X, and $q_1 + \ldots + q_m \neq 0$.) For every $i \in \omega$, the i-degree of a monomial $p = a \cdot x_1^{n_1} \cdot x_2^{n_2} \cdot \ldots x_k^{n_k}$ is defined as $\sum_{x_r \in X_i} n_r$. A rational function $u \in F(X)$ is called i-homogeneous of degree 0 if all monomials appearing in the (quotient) expression of u have the same i-degree. Let K be the subfield of $F(X)$ consisting of all rational functions in $F(X)$ that are i-homogeneous of degree 0 for all $i \in \omega$. Then $F(X)$ is a vector space over K.

For each $i \in \omega$, let V_i be the subspace of $F(X)$ which is generated by X_i, i.e. V_i is the linear span $\langle X_i\rangle$. Note that for each $i \in \omega$, V_i is finite-dimensional. Indeed, let x be any element of X_i. Then for all $y \in X_i$ with $y \neq x$, we have that $y = \frac{y}{x} \cdot x$ and $\frac{y}{x} \in K$. It follows that $V_i = \langle x\rangle$.

Let V be the weak direct product of the V_i's, i.e.,

$$V = \{f \in \prod_{i\in\omega} V_i : f \text{ has finite support, i.e., } |\{i \in \omega : f(i) \neq 0\}| < \aleph_0\}$$

with pointwise operations. Then V is an infinite-dimensional vector space over K. By our assumption, we have that V has a denumerable linearly independent subset, say $D = \{f_n : n \in \omega\}$. Then $f_n \neq 0$ for all $n \in \omega$. Using the fact that D is linearly independent, the fact that each V_i is finite-dimensional, the form of the elements of each V_i, and finally the fact that every function $f \in V$ has finite support, we may construct via mathematical induction an infinite subfamily $\mathcal{Y}$ of $\mathcal{X}$ with a multiple choice function. This completes the proof of the implication and of the theorem. □

Corollary 6.2 $\mathsf{AC}^{\aleph_0}$ *implies "every infinite-dimensional Banach space has a denumerable linearly independent subset".*

Theorem 6.3 *(i) "Every infinite-dimensional Banach space has a denumerable linearly independent subset" implies* $\mathsf{PKW}^{\aleph_0}_{\mathsf{fin},\geq 2}$ *implies "*$\forall n \in \omega\backslash\{0, 1\}$, $\mathsf{PAC}^{\aleph_0}_{\leq \mathsf{n}}$*".*

(ii) "Every infinite-dimensional Banach space has a denumerable linearly independent subset" implies "there are no amorphous sets".

(iii) "Every infinite-dimensional Banach space has a denumerable linearly independent subset" $+\mathsf{AC}^{\aleph_0}_{\mathsf{fin}}$ *implies* DF=F.

(iv) MC *does not imply "every infinite-dimensional Banach space has a denumerable linearly independent subset" in* ZFA.

P r o o f . (i) For the first implication, assume the hypothesis and let $\mathcal{A} = \{A_i : i \in \omega\}$ be a denumerable disjoint family of finite sets, each having at least two elements. Towards a proof by contradiction assume that $\mathcal{A}$ has no partial Kinna-Wagner selection function. Consider the infinite-dimensional Hilbert space $\ell^2(A)$, where $A = \bigcup \mathcal{A}$. Since $\mathcal{A}$ has no partial Kinna-Wagner function, it follows (by the definition of $\ell^2(A)$) that for all $f \in \ell^2(A)$, the support $s(f) = \{x \in A : f(x) \neq 0\}$ of f is finite. Let $Y = \{f \in \ell^2(A) : \forall i \in \omega, \sum_{x\in A_i} f(x) = 0\}$. Then Y has the following properties:

1. Y is an infinite-dimensional subspace of $\ell^2(A)$. (This is clear.)
2. Y is complete, thus a Banach space. Indeed, let $(f_n)_{n\in\omega}$ be a Cauchy sequence of elements of Y. Due to the property of the elements of Y and the fact that for all $n \in \omega$, $|s(f_n)| < \aleph_0$, we may conclude that there is an $n_0 \in \omega$ such that for all $n \in \omega$, all $i > n_0$, and all $x \in A_i$, $f_n(x) = 0$; otherwise, via an easy mathematical induction, and noting that if g is an element of $\ell^2(A)$ which is not identically zero on A_i, then there are elements $x \in A_i$ such that $g(x) > 0$ and elements $y \in A_i$ such that $g(y) < 0$, we may construct a partial Kinna-Wagner function of $\mathcal{A}$, which contradicts our assumption on $\mathcal{A}$. Thus, $(f_n)_{n\in\omega}$ can be viewed as a sequence of elements of $\mathbb{R}^{\bigcup\{A_i : i < n_0+1\}}$.
   Let $\varepsilon > 0$. Since $(f_n)_{n\in\omega}$ is Cauchy, there is an $n_0 \in \omega$ such that for all $n, m \geq n_0$, $||f_n - f_m|| < \varepsilon$, or equivalently
   $$\forall n, m \geq n_0,\ \sqrt{\sum_{x\in A} |f_n(x) - f_m(x)|^2} < \varepsilon,$$
   or equivalently
   $$\forall n, m \geq n_0,\ \sqrt{\sum_{x\in\bigcup\{A_i:i<n_0+1\}} |f_n(x) - f_m(x)|^2} < \varepsilon.$$
   Then for all $n, m \geq n_0$ and all $x \in \bigcup\{A_i : i < n_0 + 1\}$, we have that
   $$|f_n(x) - f_m(x)| \leq \sqrt{\sum_{y\in\bigcup\{A_i:i<n_0+1\}} |f_n(y) - f_m(y)|^2} < \varepsilon,$$
   which implies that for all $x \in \bigcup\{A_i : i < n_0 + 1\}$, $(f_n(x))_{n\in\omega}$ is a Cauchy sequence of reals and thus $(f_n(x))_{n\in\omega}$ converges in $\mathbb{R}$ for every $x \in \bigcup\{A_i : i < n_0 + 1\}$. Define a mapping $f : A \to \mathbb{R}$ by requiring

$f(x) = \lim_{n\to\infty} f_n(x)$ if $x \in \bigcup\{A_i : i < n_0 + 1\}$ and $f(x) = 0$ if $x \in \bigcup\{A_i : i \geq n_0 + 1\}$. Since for all $n \in \omega$, $\sum_{x\in A_i} f_n(x) = 0$, it follows that $\sum_{x\in\bigcup\{A_i : i<n_0+1\}} f(x) = 0$. Thus, $f \in Y$. Furthermore, it is easy to show that $(f_n)_{n\in\omega}$ converges to f, so we leave the details to the interested reader. Thus, Y is a Banach space.

By our hypothesis, there is a denumerable linearly independent subset $D = \{f_n : n \in \omega\}$ of Y. Taking into account,

(a) the definition of Y,
(b) the fact that no f_n can be the constant function **0** (i.e., the identically zero function),
(c) D is not included in the finite-dimensional subspace $Z_n = \{f \in Y : \forall k > n, \forall x \in A_k, f(x) = 0\}$ of Y for each $n \in \omega$,

we may easily define a partial Kinna-Wagner selection function for the family $\mathcal{A}$, which is a contradiction. This completes the proof of the first implication.

The second implication can be proved via mathematical induction.

(ii) Assume the hypothesis and, towards a proof by contradiction, let X be an amorphous set (i.e., X is infinite and cannot be expressed as a disjoint union of two infinite sets). Consider the infinite-dimensional Hilbert space $\ell^2(X)$. By our hypothesis, $\ell^2(X)$ has a denumerable linearly independent subset, say $D = \{f_n : n \in \omega\}$. Since X is amorphous, we have that for all $n \in \omega$, the support $s(f_n)$ of f_n is finite, and since D is linearly independent, we must have that $\bigcup\{s(f_n) : n \in \omega\}$ is infinite (otherwise, D can be viewed as a subset of the finite-dimensional space $\mathbb{R}^{\bigcup\{s(f_n):n\in\omega\}}$, which is impossible), thus $\mathcal{S} = \{s(f_n) : n \in \omega\}$ is countably infinite. Without loss of generality assume that $\mathcal{S}$ is disjoint; otherwise, using the fact that $\mathcal{S}$ is denumerable and that $s(f_n)$ is finite for every $n \in \omega$, we may easily construct a denumerable disjoint family $\{d_{n_i} : i \in \omega\}$ (where $(n_i)_{i\in\omega}$ is a strictly increasing sequence) such that for all $i \in \omega$, $d_{n_i} \subseteq s(f_{n_i})$. But then $\{\bigcup\{s(f_{2n}) : n \in \omega\}, X\setminus\bigcup\{s(f_{2n}) : n \in \omega\}\}$ is a partition of X into two infinite sets, which is a contradiction. This completes the proof.

(iii) Let A be an infinite Dedekind-finite set and let $H = \ell^2(A)$. For each $x \in H$, $s(x) = \{a \in A : x(a) \neq 0\}$ is finite. Let $\{x_n : n \in \omega\}$ be a denumerable linearly independent subset of H. For each $k \in \omega$ there is an $n \in \omega$ such that $s(x_n)\setminus\bigcup_{i\leq k} s(x_i) \neq \varnothing$. (Because $\{x \in H : s(x) \subseteq \bigcup_{i\leq k} s(i)\}$ is finite-dimensional.) We define a subsequence $\{y_j\}_{j=0}^{\infty}$ of $\{x_n\}_{n=0}^{\infty}$ by recursion as follows: $y_0 = x_0$ and $y_{j+1} = x_m$ where m is the least natural number k for which $s(x_k)\setminus\bigcup_{i<j} s(y_i) \neq \varnothing$. Then the sequence $(A_n)_{n\in\omega}$ defined by $A_0 = s(y_0)$ and for $n > 0$, $A_n = s(y_n)\setminus s(y_{n-1})$, is a sequence of non-empty, pairwise disjoint, finite subsets of A. Applying $\mathsf{AC}^{\aleph_0}_{\mathsf{fin}}$ gives a denumerable subset of A. This is a contradiction. Thus, DF=F holds.

(iv) This follows from part (i) and the fact that the second Fraenkel model $\mathcal{N}2$ in [6] satisfies $\mathsf{MC} + \neg(\forall n \in \omega\setminus\{0,1\}, \mathsf{PAC}^{\aleph_0}_{\leq \mathsf{n}})$ (cf. [6]). □

We note that the reason one can not work directly with the space $\ell^2(A)$ in the proof of the first implication in part (i) of Theorem 6.3, but has to consider a suitable subspace of $\ell^2(A)$, is that $\ell^2(A)$ does have a denumerable linearly independent subset, namely $\{\chi_{A_n} : n \in \omega\}$ where χ_{A_n} is the characteristic function of A_n.

We should also like to point out here that the statement "every infinite-dimensional separable normed space X has a denumerable linearly independent subset which is dense in X" is provable in ZF.

Indeed, let X be an infinite-dimensional and separable normed space. Let $D = \{d_n : n \in \omega\}$ be a denumerable dense subset of X. Then the topology on X induced by the norm has a countable base, say $\mathcal{O} = \{O_n : n \in \omega\}$. Via mathematical induction we construct the required set. Start with any element $x_0 \in O_0$. Assume that we have chosen elements $x_0, x_1, \ldots, x_n \in X$ such that for $1 \leq i \leq n$, $x_i \in O_i\setminus\langle x_0, \ldots, x_{i-1}\rangle$. By Lemma 3.5 we have that $O_{n+1}\setminus\langle x_0, x_1, \ldots, x_n\rangle$ is non-empty and open. Let $m_{n+1} = \min\{m \in \omega : d_m \in O_{n+1}\setminus\langle x_0, x_1, \ldots, x_n\rangle\}$. Put $x_{n+1} = d_{m_{n+1}}$. This concludes the inductive step.

By the inductive construction, it follows that $\{x_n : n \in \omega\}$ is a denumerable linearly independent subset of X which is dense in X.

Furthermore, it is interesting to note that the statement "no infinite-dimensional separable Banach space has a Hamel basis which can be written as a denumerable union of finite sets" is also provable in ZF. Indeed, if X is an infinite-dimensional separable Banach space, then $|X| = 2^{\aleph_0}$. Thus, if X had a Hamel basis $B = \bigcup\{B_n : n \in \omega\}$, $|B_n| < \aleph_0$ for all $n \in \omega$, then $|B| = \aleph_0$. This would contradict the result of Theorem 4.3 that no infinite-dimensional Banach space has a denumerable Hamel basis.

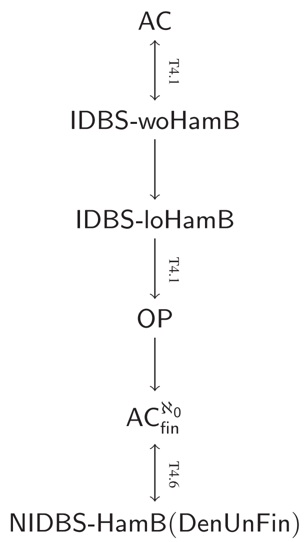

Fig. 1 Results from § 4.

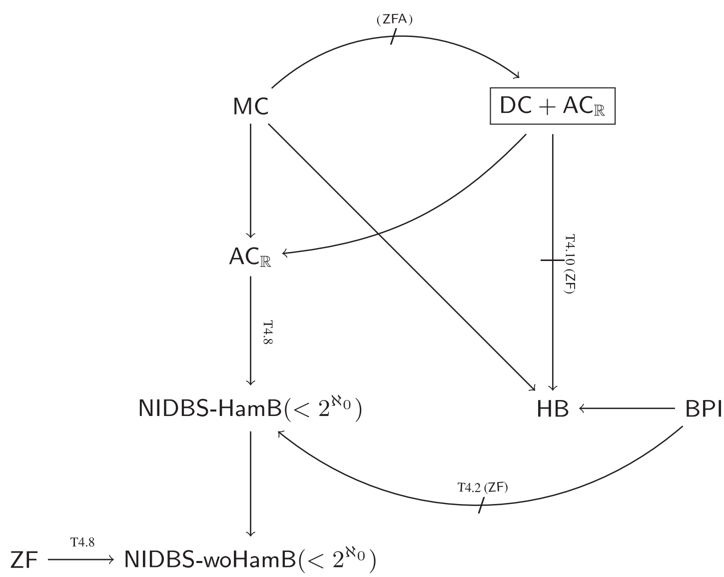

Fig. 2 Results from § 4.

We do not know whether "no infinite-dimensional separable Banach space has a well-orderable Hamel basis of cardinality $< 2^{\aleph_0}$" is provable in ZF.

Theorem 6.4 *(i) "Every infinite-dimensional Banach space X has a linearly independent subset of cardinality $\geq 2^{\aleph_0}$" implies* DF=F.

(ii) "No infinite-dimensional Banach space has a Hamel basis of cardinality $< 2^{\aleph_0}$" does not imply "every infinite-dimensional Banach space has a linearly independent subset of cardinality $\geq 2^{\aleph_0}$" in ZFA.

(iii) Assume $\mathsf{CH} + \mathsf{W}_{\aleph_2}$. *Then every infinite-dimensional Banach space has a linearly independent subset of cardinality $\geq 2^{\aleph_0}$.*

(iv) "Every infinite-dimensional Banach space has a linearly independent subset of cardinality $\geq 2^{\aleph_0}$" does not imply $\mathsf{AC}^{\aleph_0}$ *in* ZFA.

Proof. (i) Assume the hypothesis. By Claim 5.3 of the proof of Theorem 5.2 we have that our hypothesis implies "no infinite-dimensional Banach space can be written as a denumerable union of finite-dimensional subspaces". By Theorem 5.1, the latter statement implies $\mathsf{AC}^{\aleph_0}_{\mathsf{fin}}$. Since our hypothesis clearly implies that every

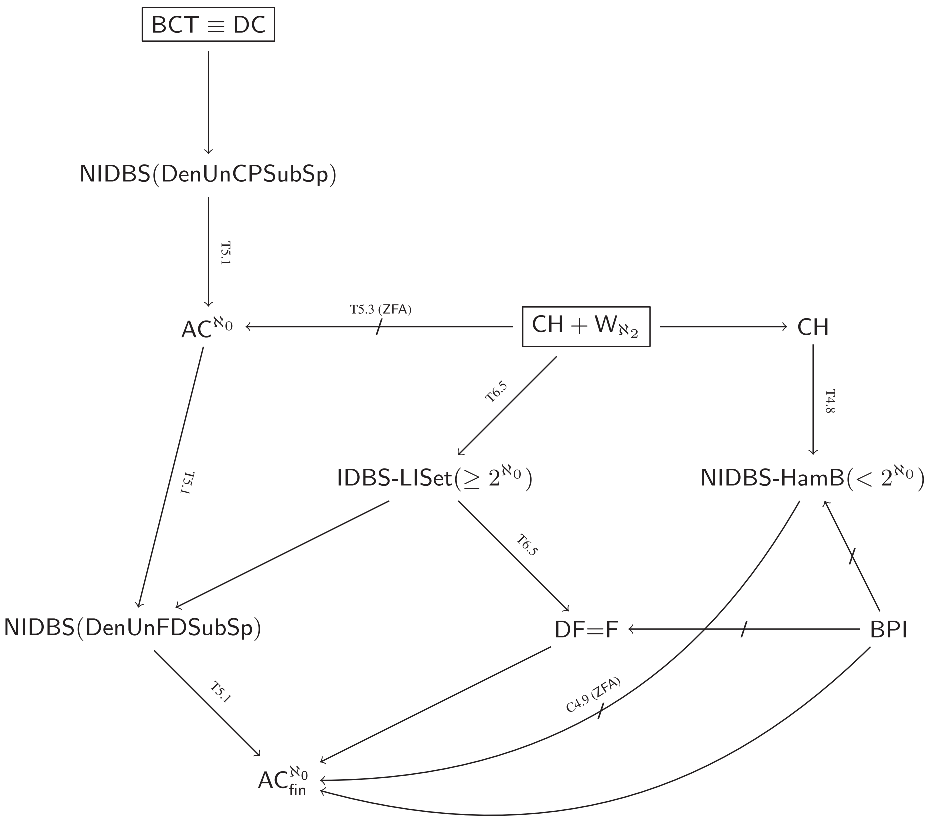

Fig. 3 Results from §§ 5 and 6.

infinite-dimensional Banach space has a denumerable linearly independent subset, the conclusion follows from part (iii) of Theorem 6.3.

(ii) This follows from Corollary 4.9(ii) and Theorem 6.3(iv) (or from part (i) of the current theorem and the fact that DF=F is false in several Fraenkel-Mostowski models, e.g., the second Fraenkel model $\mathcal{N}2$ in [6]).

(iii) Assume the hypothesis and let X be an infinite-dimensional Banach space. By $\mathsf{W}_{\aleph_2}$, either $|X| \leq \aleph_2$ or $|X| \geq \aleph_2$. In the first case, X is well-orderable, say by $\leq$. Choose a non-zero element x_0 of X and define a function f from $\aleph_1 \to X$ by recursion as follows. $f(0) = x_0$ and

$$f(j) = \text{the } \leq\text{-least element of } X \text{ which does not belong to the linear span of } \{x_i : i < j\}. \qquad (11)$$

$f(j)$ is defined for every $j < \aleph_1$; first note that the recursion can not stop after finitely many steps, since X is infinite-dimensional. Furthermore, the recursion can not be terminated at some countable stage. If not, let j^* be the least ordinal in $\aleph_1$ such that $X = \langle\{x_i : i < j^*\}\rangle$. By the construction, $\{x_i : i < j^*\}$ is linearly independent, and since it spans X, it is also a Hamel basis for X. Since $|\{x_i : i < j^*\}| = \aleph_0$, this contradicts the result of Theorem 4.3 that (in ZF) no infinite-dimensional Banach space has a denumerable Hamel basis.

Thus, the range of f will be an $\aleph_1$-sized linearly independent subset of X and by the assumption of CH we have that X has a continuum sized linearly independent subset.

In the second case, if $|X| \geq \aleph_2$, then X has a well-orderable subset Y such that $|Y| = \aleph_2$. Then the subspace $\langle Y\rangle$ of X spanned by Y is well-orderable, in particular, $|\langle Y\rangle| = \aleph_2$ for $|Y| = \aleph_2$ and $2^{\aleph_0} = \aleph_1$. Further, $\langle Y\rangle$ is (not necessarily a Banach space and) not finite-dimensional, since any finite-dimensional, non-trivial, real vector space has cardinality $2^{\aleph_0}$ and according to our hypothesis $2^{\aleph_0} = \aleph_1$. Let $\leq$ be a well ordering of $\langle Y\rangle$ and let x_0 be an element of Y. Define $f : \aleph_1 \to \langle Y\rangle$ as was done previously in (11). Then once again $f(j)$ is defined for every $j \in \aleph_1$ for, if not, the set $\{x_i : i < j^*\}$ spans $\langle Y\rangle$ for some $j^* < \aleph_1$. But then $|\langle Y\rangle| = |\langle\{x_i : i < j^*\}\rangle| = 2^{\aleph_0} = \aleph_1$ and $|\langle Y\rangle| = \aleph_2$, a contradiction. It follows that the range of f is a continuum sized linearly independent subset of X. This completes the proof of part (iii).

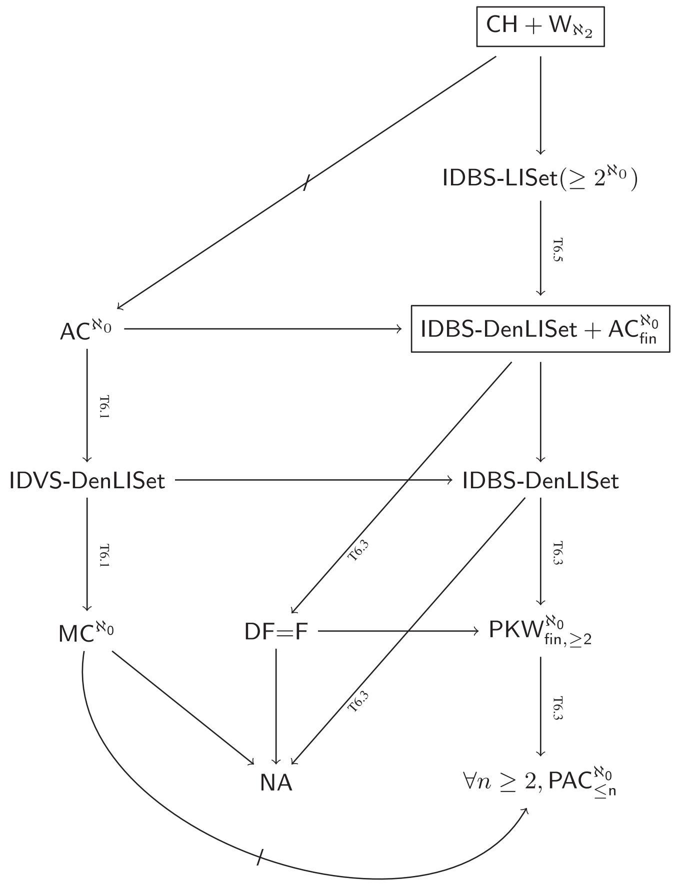

Fig. 4 Results from § 6.

(iv) We may work here as in the proof of Theorem 5.2. In particular, we use the permutation model $\mathcal{N}16$ in [6] (which satisfies $W_{\aleph_n}$ for all $n \in \omega$), but starting with a ground model $\mathcal{M}$ such that $\mathcal{M} \models \mathsf{ZFA} + \mathsf{AC} + \mathsf{CH}$. The result now follows from the properties of the FM model and part (iii) of the current theorem. □

We should like to point out here that in view of Theorem 4.8 and the fact that AC is equivalent to the trichotomy of cardinals, i.e., the statement "for all sets A and B, $|A| \leq |B|$ or $|B| \leq |A|$" (cf. [6]), we obtain that "for every infinite-dimensional Banach space X and for every Hamel basis B of X, $2^{\aleph_0} \leq |B|$" is a theorem of ZFC. However, the latter proposition is not provable in ZF. Indeed, we observe that it implies DF=F; assuming the above proposition, let D be an infinite Dedekind-finite set and let X be the infinite-dimensional Hilbert space $\ell^2(D)$. Then for each $f \in X$, the support $s(f)$ of f is finite. Thus, $B = \{\chi_{\{d\}} : d \in D\}$ is a Hamel basis for X. By assumption, we have that $2^{\aleph_0} \leq |B|$, from which it follows that D has a denumerable subset, a contradiction.

7 Summary

We summarize main results of our paper in the form of diagrams. In order to facilitate the reader in studying the diagrams, we introduce some further notation for propositions whose mutual relationships and set-theoretic strength were studied in the paper.

1. NIDBS-HamB($\aleph_0$): No infinite-dimensional Banach space has a denumerable Hamel basis.
2. NIDBS-HamB($< 2^{\aleph_0}$): No infinite-dimensional Banach space has a Hamel basis of cardinality $< 2^{\aleph_0}$.
3. NIDBS-woHamB($< 2^{\aleph_0}$): No infinite-dimensional Banach space has a well-orderable Hamel basis of cardinality $< 2^{\aleph_0}$.

4. NIDBS-HamB(DenUnFin): No infinite-dimensional Banach space has a Hamel basis which can be written as a denumerable union of finite sets.
5. IDBS-loHamB: Every infinite-dimensional Banach space has a linearly orderable Hamel basis.
6. IDBS-woHamB: Every infinite-dimensional Banach space has a well-orderable Hamel basis.
7. ML: Mazur's Lemma; Let X be an infinite-dimensional Banach space, let Y be a finite-dimensional vector subspace of X, and let $\varepsilon > 0$. Then there is a unit vector $x \in X$ such that $||y|| \leq (1+\varepsilon)||y+\alpha x||$ for all $y \in Y$ and all scalars α).
8. NIDBS(DenUnCPSubSp): No infinite-dimensional Banach space can be written as a denumerable union of closed proper subspaces.
9. NIDBS(DenUnFDSubSp): No infinite-dimensional Banach space can be written as a denumerable union of finite-dimensional subspaces.
10. Let F be a field. IDVS-DenLISet(F): every infinite-dimensional vector space over F has a denumerable linearly independent subset.
11. IDVS-DenLISet: $(\forall F)$(IDVS-DenLISet(F)), where the parameter F denotes a field.
12. IDBS-LISet($\geq 2^{\aleph_0}$): every infinite-dimensional Banach space has a linearly independent subset of cardinality $\geq 2^{\aleph_0}$.
13. IDBS-DenLISet: every infinite-dimensional Banach space has a denumerable linearly independent subset.
14. NA: There are no amorphous sets.

We first list main results of the paper that are proven to be theorems of ZF:

(ZF) A normed real vector space $(X, ||\cdot||)$ is finite-dimensional if and only if its closed unit ball B_X is compact (Theorem 3.6).
(ZF) Mazur's Lemma ML (Lemma 3.7).
(ZF) NIDBS-woHamB($< 2^{\aleph_0}$) (Theorem 4.8(ii)).

We proceed now with the diagrams that summarize results of our paper. Figures 1, 3 and 4 suggest several open problems (and we note that there are no open problems from Figure 2). To mention a few:

(a) Does IDBS-DenLISet imply DF=F? (Cf. Figure 4),
(b) does $\mathrm{AC}^{\aleph_0}$ imply IDBS-LISet($\geq 2^{\aleph_0}$)? (Cf. Figures 3 & 4),
(c) is NIDBS(DenUnCPSubSp) equivalent to DC? (Cf. Figure 3),
(d) does $\mathrm{AC}^{\aleph_0}$ imply NIDBS(DenUnCPSubSp)? (Cf. Figure 3),
(e) does $\mathrm{AC}^{\aleph_0}_{\mathrm{fin}}$ imply NIDBS(DenUnFDSubSp)? (Cf. Figure 3),
(f) does IDBS-LISet($\geq 2^{\aleph_0}$) imply NIDBS-HamB($< 2^{\aleph_0}$)? (Cf. Figure 3).

Another interesting open problem which is related to problem (f) above, as well as to Lemma 4.7 & Theorem 4.8, is the following:

(g) If B is a Dedekind-infinite Hamel basis of a Banach space such that $|B| \leq 2^{\aleph_0}$, is it true (in ZF) that $|B| = 2^{\aleph_0}$?
(h) Does IDBS-loHamB imply IDBS-woHamB? (Cf. Figure 1)
(i) Does OP imply IDBS-loHamB? (Cf. Figure 1)

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