Mathematics 131B, Fall 2016 – ℝami Luisto
Homework 5 - Due date Friday October 28th.

You are welcomed, even encouraged to discuss the homework problems together. If and when you do you should, however, acknowledge your collaboration and mention who you have been working with in your returned solutions. I sincerely hope that it goes without saying that copying someone else's work without thought is not 'working together', and while not technically disallowed, is very counterproductive for your learning process. While in certain previous courses one might get a good grade just my memorizing a few solution algorithms, many problems in this course require some level of understanding what is going on mathematically. This will be the rule rather than the exception in all further courses, and trying to mimic proofs without thinking will lead to misery after the first few weeks. Understanding what is happening, on the other hand, will lead to feelings of **great beauty** and **elegance**.

If you are having problems with an exercise, you can and should ask for help. However, if you glance at a problem and immediately shout "I don't know how to solve this!", you are not having problems with the exercise but rather you are having problems with not expending any effort to solve the problem. **"You never call any question impossible, said Harry, until you have taken an actual clock and thought about it for five minutes, by the motion of the minute hand. Not five minutes metaphorically, five minutes by a physical clock."** (*Harry Potter in Harry Potter and the Methods of Rationality.*) `\end{rant}`

Prepping problems. These problems will not be graded. They are voluntary, very basic problems that can help you get up to speed.

p1) Remind yourself how the Riemann integral was defined.
p2) Recall how the convergence of a series of real numbers was defined.

Homework problems

(1) Let $f_n \colon \mathbb{R} \to \mathbb{R}$ be a sequence of continuous functions converging uniformly to $f \colon \mathbb{R} \to \mathbb{R}$. Suppose f_n are all *bounded functions.* (A function $g \colon \mathbb{R} \to \mathbb{R}$ is bounded if there exists a number $M \geq 0$ such that $|g(x)| \leq M$ for all $x \in \mathbb{R}$.) Show that the limit f is bounded as well.
(2) Give an example of a sequence of bounded functions $\mathbb{R} \to \mathbb{R}$ converging pointwise to an unbounded function.
(3) Suppose (f_n) and (g_n) are sequences of continuous functions $\mathbb{R} \to \mathbb{R}$, with $f_n \to f$ and $g_n \to g$ uniformly.
   (a) Show that the mappings $h_n \colon \mathbb{R} \to \mathbb{R}$, $h_n(x) = f_n(x) + g_n(x)$ converge uniformly to the mapping $f + g$.
   (b) Suppose that the mappings f_n and g_n are defined on the open unit interval and are converging uniformly to mappings f and

g, respectively. Does $f_n \cdot g_n$ necessarily converge uniformly to $f \cdot g$?

(4) Use the Weierstrass M-test (this will be studied on Monday lecture) to show that the series $\sum_{n=1}^{\infty} 4^{-n} \cos(32^n \pi x)$ converges uniformly to a continuous function $f \colon \mathbb{R} \to \mathbb{R}$.

The following is an extra problem to those who want to learn something more. It will not be graded.

Extra challenge: Take the limit function f from exercise (4) and show that for every $j \in \mathbb{Z}$ and $m \in \mathbb{N}$ we have

$$\left| f\left(\frac{j+1}{32^m}\right) - f\left(\frac{j}{32^m}\right) \right| \geq 4^{-m}. \tag{P}$$

(Hint: use the identity

$$\sum_{n=1}^{\infty} a_n = \left(\sum_{n=1}^{m-1} a_n\right) + a_m + \sum_{n=m+1}^{\infty} a_n$$

for a suitable sequence a_n. Recall also that cos is peridic with period 2π and that $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ for any r with $|r| < 1$. Finally you need that cos is 1-Lipschitz, which follows from the mean value theorem.)

Using this property (P) show that f is not differentiable at any given point $x_0 \in \mathbb{R}$! (Hint: for all $x_0 \in \mathbb{R}$ and $m \in \mathbb{Z}$, there exists $j \in \mathbb{Z}$ such that $j \leq 32^m x_0 \leq j+1$.)