Mathematics 131B, Fall 2016 – $\mathbb{R}$ami Luisto
Homework 4 - Due date Friday October 21st.

You are welcomed, even encouraged to discuss the homework problems together. If and when you do you should, however, acknowledge your collaboration and mention who you have been working with in your returned solutions. I sincerely hope that it goes without saying that copying someone else's work without thought is not 'working together', and while not technically disallowed, is very counterproductive for your learning process. While in certain previous courses one might get a good grade just my memorizing a few solution algorithms, many problems in this course require some level of understanding what is going on mathematically. This will be the rule rather than the exception in all further courses, and trying to mimic proofs without thinking will lead to misery after the first few weeks. Understanding what is happening, on the other hand, will lead to feelings of **great beauty** and **elegance**.

If you are having problems with an exercise, you can and should ask for help. However, if you glance at a problem and immediately shout "I don't know how to solve this!", you are not having problems with the exercise but rather you are having problems with not expending any effort to solve the problem. **"You never call any question impossible, said Harry, until you have taken an actual clock and thought about it for five minutes, by the motion of the minute hand. Not five minutes metaphorically, five minutes by a physical clock."** (*Harry Potter in Harry Potter and the Methods of Rationality.*) `\end{rant}`

This homework set has fewer problems than usually to give you time to study for midterm. If you feel that you did not do well on certain questions in the midterm, you may take the questions as an extra problems of this homework set for a chance to earn extra credit for the midterm. (I.e. if you feel that on Midterm's question 2 you got a bad solution, you may take Q2 as an extra problem. A correct solution will earn you some extra credit counted towards the score of that particular problem on the effect of the midterm on your grade.)

Prepping problems. These problems will not be graded. They are voluntary, very basic problems that can help you get up to speed.

p1) Show that an open ball is always an open set in a metric space.

p2) Show that $\complement f^{-1}V = f^{-1}\complement U$ and conclude that for a continuous maps the pre-images of closed sets are closed.

Homework problems

(1) We say that a mapping $f\colon (X,d) \to (Y,d')$ is *L-Lipschitz*, if for all $x, y \in X$,

$$d'(f(x), f(y)) \leq L d(x,y).$$

Show that an L-Lipschitz mapping is continuous. (Hint: use the ε-δ - defintion.)

(2) Let (X, d) and (Y, d') be metric spaces.
  (a) Suppose $A \subset X$ is a bounded set and $f\colon X \to Y$ an L-Lipschitz map. Show that $fA \subset Y$ is bounded.
  (b) Suppose $K \subset X$ is a compact set and $f\colon X \to Y$ a continuous map. Show that $fK \subset Y$ is bounded.

(3) Let (x_n) be a Cauchy sequence in (X, d) and $f\colon X \to Y$ an L-Lipschitz map with (Y, d') a metric space. Show that the sequence (y_n), $y_n = f(x_n)$ for all $n \in \mathbb{N}$ is Cauchy.

The following is an extra problem to those who want to learn something more. It will not be graded.

Extra challenge: Let X be a complete metric space, and $f\colon X \to X$ an L-Lipschitz map with $L < 1$. (Such a mapping is called a *contraction.*) Show that there exists a unique point $x_0 \in X$ such that $f(x_0) = x_0$. Such a point is called a *fixed point of f.*

This result is called the *Banach fixed point theorem.*

(Hint: Show first that a fixed point must be unique by using the fact that f is a contraction. To find a fixed point, take any point $a \in X$ and costruct a sequence by setting $x_0 = a$, and $x_{k+1} = f(x_k)$ for all $k \in \mathbb{N}$. Show by induction that $d(x_{k+1}, x_k) \leq L^k d(x_1, x_0)$ and deduce that (x_k) is Cauchy by using triangle inequality and the sum of geometric series.)