Mathematics 131B, Fall 2016 – ℝami Luisto
Homework 3 - Due date Friday October 14th.

You are welcomed, even encouraged to discuss the homework problems together. If and when you do you should, however, acknowledge your collaboration and mention who you have been working with in your returned solutions. I sincerely hope that it goes without saying that copying someone else's work without thought is not 'working together', and while not technically disallowed, is very counterproductive for your learning process. While in certain previous courses one might get a good grade just my memorizing a few solution algorithms, many problems in this course require some level of understanding what is going on mathematically. This will be the rule rather than the exception in all further courses, and trying to mimic proofs without thinking will lead to misery after the first few weeks. Understanding what is happening, on the other hand, will lead to feelings of **great beauty** and **elegance**.

If you are having problems with an exercise, you can and should ask for help. However, if you glance at a problem and immediately shout "I don't know how to solve this!", you are not having problems with the exercise but rather you are having problems with not expending any effort to solve the problem. **"You never call any question impossible, said Harry, until you have taken an actual clock and thought about it for five minutes, by the motion of the minute hand. Not five minutes metaphorically, five minutes by a physical clock."** (*Harry Potter in Harry Potter and the Methods of Rationality.*) `\end{rant}`

Prepping problems. These problems will not be graded. They are voluntary, very basic problems that can help you get up to speed.

p1) Show that a compact metric space is complete.
p2) Show that a finite metric space is always both complete and compact.
p3) Show that $(\mathbb{R}, d_e)$ is not a compact metric space.

In this problem set, by compact we mean sequentially compact!

Homework problems

1. Show that the metric space (X, d_{disc}), where d_{disc} is the discrete metric, is compact if and only if X is a finite set.
2. Show that a compact set K is always *bounded*, i.e. there exists a point x and a radius r such that the $K \subset B(x, r)$.
3. Let K be a compact subset of a metric space (X, d) and $x_0 \in X$. Show that there exists a point $y_0 \in K$ such that $d(x_0, y_0) = d(x_0, K)$, where
$$d(x_0, K) = \inf\{d(x_0, y) \mid y \in K\}.$$

4. Let A and B be a compact subsets of a metric space (X, d). Show that there exists points $a_0 \in A$ and $b_0 \in B$ such that $d(a_0, b_0) = d(A, B)$, where
$$d(A, B) = \inf\{d(x, y) \mid x \in A, y \in B\}.$$
5. Show that the space $(C([0, 1], \mathbb{R}), d_\infty)$ is not compact.
6. Let $C \geq 0$. Show that the set
$$\{f \in C([0, 1], \mathbb{R}) \mid f(x) \leq C \text{ for all } x \in [0, 1]\}$$
is not compact subset of the space $(C([0, 1], \mathbb{R}), d_\infty)$.

The following is an extra problem to those who want to learn something more. It will not be graded.

Extra challenge 1: This extra challenge continues problems 3 and 4. Show first that the function $(A, B) \mapsto d(A, B)$ is not a metric in the collection of compact subsets of X.

Let (X, d) be a metric space and $\mathcal{K}(X)$ the collection of all compact subsets of X. Define for $A, B \in \mathcal{K}(X)$
$$d_{\mathcal{H}}(A, B) = \inf\{r > 0 \mid A \subset B_X(B, r) \text{ and } B \subset B_X(A, r)\}$$
where
$$B_X(A, r) = \{y \in X \mid d(y, A) < r\}.$$
Show that $d_{\mathcal{H}}$ is a metric in $\mathcal{K}(X)$. (This metric is called *the Hausdorff metric.*)