Mathematics 131B, Fall 2016 – ℝami Luisto
Homework 2 - Due date Friday October 7th.

You are welcomed, even encouraged to discuss the homework problems together. If and when you do you should, however, acknowledge your collaboration and mention who you have been working with in your returned solutions. I sincerely hope that it goes without saying that copying someone else's work without thought is not 'working together', and while not technically disallowed, is very counterproductive for your learning process. While in certain previous courses one might get a good grade just my memorizing a few solution algorithms, many problems in this course require some level of understanding what is going on mathematically. This will be the rule rather than the exception in all further courses, and trying to mimic proofs without thinking will lead to misery after the first few weeks. Understanding what is happening, on the other hand, will lead to feelings of **great beauty** and **elegance**.

If you are having problems with an exercise, you can and should ask for help. However, if you glance at a problem and immediately shout "I don't know how to solve this!", you are not having problems with the exercise but rather you are having problems with not expending any effort to solve the problem. **"You never call any question impossible, said Harry, until you have taken an actual clock and thought about it for five minutes, by the motion of the minute hand. Not five minutes metaphorically, five minutes by a physical clock."** (*Harry Potter in Harry Potter and the Methods of Rationality.*) `\end{rant}`

Prepping problems. These problems will not be graded. They are voluntary, very basic problems that can help you get up to speed.

p1) Can the intersection of two metric balls be a metric ball? (Hint: Study the taxicab metric or the maximum metric in the plane.)
p2) Show that $\overline{\overline{A}} = \overline{A}$.
p3) Show that ∂A is a closed set.
p4) Convince yourself that the interior points, boundary points and exterior points of a set in a metric space X are all mutually disjoint but cover all of X.

Homework problems

1. Let (X, d) be a metric space and $x_0 \in X$. Show that $X \setminus \{x_0\}$ is an open set.
2. Equip the plane with the maximum metric:
$$d_\infty(\mathbf{x}, \mathbf{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|\}.$$
What is the closure of the set
$$A = \{(k/n, 1/n) \mid k \in \mathbb{Z}, n \in \mathbb{N}\}?$$
(Begin by drawing a picture.)
3. Let (X, d) be a metric space and (x_n) a sequence.

(a) Show that if (x_n) converges to a point x_0, then every subsequence of (x_n) also converges to the point x_0.
(b) Suppose that every subsequence of (x_n) has a subsequence converging to a_0. Show that also $x_n \to a$ as $n \to \infty$.

4. Show that $\partial(\partial U) \subset \partial U$. After this meditate on the example in Friday lecture: the boundary of $\mathbb{Q}$ in the metric space $(\mathbb{R}, d_{\mathbf{eucl.}})$.
5. Show that $\partial(\partial(\partial U)) = \partial(\partial U)$ and then deduce that

$$\underbrace{\partial \cdots \partial}_{n \text{ times}} U = \partial\partial U$$

whenever $n \geq 2$.

Hint: Prove the inclusions $\partial\partial\partial U \subset \partial\partial U$ and $\partial\partial\partial U \supset \partial\partial U$. The first one is a direct-ish calculation similar to problem 4. For the second one recall that $\complement\partial A = (\operatorname{int} A) \cup (\operatorname{ext} A)$, which is an open set.

The following is an extra problem to those who want to learn something more. It will not be graded.

Extra challenge 1: Let $C([0,1], \mathbb{R})$ the space of continuous functions $f\colon [0,1] \to \mathbb{R}$ equipped with the supremum metric:

$$d_\infty(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|.$$

Take

$$U = \{f \in C([0,1], \mathbb{R}) \mid f(x) > 0 \text{ for all } x \in [0,1]\}.$$

Is U open in $(C([0,1], \mathbb{R}), d_\infty)$? What is the boundary ∂U?

Extra challenge 2: Let $C([0,1], \mathbb{R})$ the space of continuous functions $f\colon [0,1] \to \mathbb{R}$ equipped with the L_1-metric:

$$d_1(f, g) = \int_0^1 |f(x) - g(x)| \, \mathrm{d}\, x$$

(You may assume d_1 is a metric in $C([0,1], \mathbb{R})$.)

Let

$$U = \{f \in C([0,1], \mathbb{R}) \mid f(x) > 0 \text{ for all } x \in [0,1]\}.$$

Is U open in $(C([0,1], \mathbb{R}), d_1)$? What is the boundary ∂U?