Mathematics 131B, Fall 2016 – ℝami Luisto
Homework 1 - Due date Friday September 30th.

You are welcomed, even encouraged to discuss the homework problems together. If and when you do you should, however, acknowledge your collaboration and mention who you have been working with in your returned solutions. I sincerely hope that it goes without saying that copying someone else's work without thought is not 'working together', and while not technically disallowed, is very counterproductive for your learning process. While in certain previous courses one might get a good grade just my memorizing a few solution algorithms, many problems in this course require some level of understanding what is going on mathematically. This will be the rule rather than the exception in all further courses, and trying to mimic proofs without thinking will lead to misery after the first few weeks. Understanding what is happening, on the other hand, will lead to feelings of **great beauty** and **elegance**.

If you are having problems with an exercise, you can and should ask for help. However, if you glance at a problem and immediately shout "I don't know how to solve this!", you are not having problems with the exercise but rather you are having problems with not expending any effort to solve the problem. **"You never call any question impossible, said Harry, until you have taken an actual clock and thought about it for five minutes, by the motion of the minute hand. Not five minutes metaphorically, five minutes by a physical clock."** (*Harry Potter in Harry Potter and the Methods of Rationality.*)
`\end{rant}`

Prepping problems. These problems will not be graded. They are voluntary, very basic problems that can help you get up to speed.

p1) Show that the absolute value of the real numbers is a norm. The absolute value of real numbers is defined as

$$|a| = \begin{cases} a, & \text{if } a \geq 0, \\ -a, & \text{if } a < 0. \end{cases}$$

p2) Convince yourself that the mapping

$$d \colon X \times X \to \mathbb{R}, \quad d(x,y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y \end{cases}$$

is a metric.

p3) Exercise 1.1.3. from Tao's book (p.9). For those with differing versions of the book; this is the exercise where you have to construct a mappings that satisfy all but one of the conditions required from a metric.

Homework problems

1. Let d_1 be the taxicab metric in the plane, i.e.

$$d_1(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + |x_2 - y_2|.$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$. Show that it is a metric. (You may use the information that for real numbers a and b, $|a+b| \leq |a| + |b|$.)

2. Let d_1 be the taxicab metric in the plane as in excercise 1. Draw the unit ball $B_{d_1}(\mathbf{0}, 1)$ with respect to this metric. Compare it to the unit ball $B_{d_e}(\mathbf{0}, 1)$ given by the Euclidean metric
$$d_e(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$
and to the unit ball $B_{d_\infty}(\mathbf{0}, 1)$ given by the maximum metric
$$d_\infty(\mathbf{x}, \mathbf{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|\}.$$
3. Let $X := C([0,1], \mathbb{R})$ be the set of continuous functions $f\colon [0,1] \to \mathbb{R}$. Using the basic results of the Riemann integral, show that the mapping
$$d_1\colon X \times X \to \mathbb{R}, \quad d_1(f,g) = \int_0^1 |f(x) - g(x)| \,\mathrm{d}x$$
is a metric. (You may again use the fact that for real numbers a and b, $|a+b| \leq |a| + |b|$.)
4. Let $C([0,1], \mathbb{R})$ be the set of continuous functions $f\colon [0,1] \to \mathbb{R}$ and denote by d_1 the metric given in exercise 3 and by d_∞ the sup-metric defined in the lectures:
$$d_\infty(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)|.$$
Denote by f_0 the constant function $f_0(x) = 0$.
   (a) Show that $B_{d_\infty}(f_0, 1) \subset B_{d_1}(f_0, 1)$.
   (b) Show that $B_{d_1}(f_0, r) \not\subset B_{d_\infty}(f_0, 1)$ for all $r > 0$.
5. Let $f\colon \mathbb{R} \to \mathbb{R}^2$ be an injective mapping. Define a mapping $d_f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by setting
$$d_f(x,y) = d_e(f(x), f(y)),$$
where d_e is the Euclidean metric in the plane, i.e.
$$d_e(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$
Show that $(\mathbb{R}, d_f)$ is a metric space. (Hint: Go bravely forth and start checking the conditions of a metric. Use the fact that d_e is a metric which thus satisfies the corresponding conditions.)

The following is an extra problem to those who want to learn something more. It will not be graded.

Extra challenge: Define for each $p = 1, 2, \ldots$ a metric $d_p\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$.
$$d_\infty(\mathbf{x}, \mathbf{y}) = \left(|x_1 - y_1|^p + |x_2 - y_2|^p\right)^{1/p}.$$
(You may assume d_p is a metric.)

Show that for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, $d_p(\mathbf{x}, \mathbf{y}) \to d_\infty(\mathbf{x}, \mathbf{y})$ as $p \to \infty$ where d_∞ the maximum-metric. (This result gives one reason why the maximum-metric is denoted d_∞.)